

BEST APPROXIMATION IN MAX-PLUS SEMIMODULES

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ABSTRACT. We establish new results concerning projectors on max-plus spaces, as well as separating half-spaces, and derive an explicit formula for the distance in Hilbert’s projective metric between a point and a half-space over the max-plus semiring, as well as explicit descriptions of the set of minimizers. As a consequence, we obtain a cyclic projection type algorithm to solve systems of max-plus linear inequalities.

1. INTRODUCTION

Let \mathbb{R}_{\max} denote the so-called *max-plus algebra*, which is the semiring composed of the set $\mathbb{R} \cup \{-\infty\}$ endowed with the maximization operation as addition $\mu \oplus \nu := \max(\mu, \nu)$, the usual addition as multiplication $\mu \otimes \nu := \mu + \nu$ (also for $\mu = \nu = -\infty$), and the neutral elements $-\infty$ and 0 for addition \oplus and multiplication \otimes respectively. We shall often denote the multiplication of \mathbb{R}_{\max} by concatenation (except when the omission of the symbol \otimes leads to an ambiguity).

The space \mathbb{R}_{\max}^n of n -dimensional vectors, endowed naturally with the pointwise addition (denoted also by \oplus) and the multiplication of a vector by a scalar (denoted below by concatenation, with the scalar on the right), is a semimodule (the analogue of a module) over \mathbb{R}_{\max} . It is also endowed with the following operation \backslash which comes from the residuation of the map that multiplies a scalar by a given vector (see Section 2):

$$x \backslash y := \sup\{\lambda \in \mathbb{R}_{\max} \mid x\lambda \leq y\}, \quad (1.1)$$

where the order \leq in (1.1) is the usual partial order.

The most natural “distance” [8, 7, 13, 16] on the space \mathbb{R}_{\max}^n is the (additive analogue of) Hilbert’s projective distance d , which can be defined

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by

$$d(x, y) := ((x \setminus y) \otimes (y \setminus x))^- , \quad (1.2)$$

where the superscript $-$ means taking the usual opposite, that is,

$$\lambda^- := -\lambda \quad \forall \lambda \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}; \quad (1.3)$$

note that here $d(x, y) \in \bar{\mathbb{R}}$. When the vectors x and y have only finite entries,

$$d(x, y) = \max_{i,j \in [n]} (x_i - y_i + y_j - x_j) ,$$

where $[n] := \{1, \dots, n\}$.

The same definition can be used on any residuated idempotent semimodule, and it generalizes there the usual Hilbert projective metric considered on cones of Banach spaces [6]: if u, v are two vectors in the interior of a closed convex pointed cone C in such a space, the Hilbert projective metric is classically defined by

$$\text{Hilb}(u, v) = \min \log \left\{ \frac{\mu}{\lambda} \mid \lambda > 0, \mu > 0, \lambda u \leq v \leq \mu u \right\} , \quad (1.4)$$

where $u \leq v$ means that $v - u \in C$. When \mathbb{R}^n is thought of as the image of the interior of the standard positive cone by the map which takes the logarithm entrywise, so that $x_i = \log u_i$ and $y_i = \log v_i$, we get $d(x, y) = \text{Hilb}(u, v)$ (see [8, Section 3.3]).

If one avoids vectors with only infinite entries, then d satisfies all the properties of a projective distance, except that it may take infinite values (see Section 2).

If V is a subset of \mathbb{R}_{\max}^n , and $x \in \mathbb{R}_{\max}^n$, one defines as for a usual distance:

$$d(x, V) := \inf_{v \in V} d(x, v), \quad (1.5)$$

and we define an *element of best approximation*, or a *best approximation*, of x in V , or a *nearest point* to x in V , as an element v_0 of V such that

$$d(x, v_0) = d(x, V). \quad (1.6)$$

In the present paper we shall study the best approximation for Hilbert's projective metric in *b-complete subsemimodules* of \mathbb{R}_{\max}^n . We recall that any semimodule V over \mathbb{R}_{\max} is an idempotent monoid for its additive law, and is thus "naturally" ordered by the relation \leq defined by

$$x \leq y \Leftrightarrow x \oplus y = y, \quad (1.7)$$

which is such that the supremum coincides with the addition \oplus of the semimodule. It is said to be *b-complete* if any subset of V bounded from above has a supremum in V and if the scalar multiplication distributes over all such infinite sums (see Litvinov, Maslov and Shpiz [17]). In particular, \mathbb{R}_{\max}^n is a *b-complete semimodule* over \mathbb{R}_{\max} , and its natural order is the usual partial order. A subsemimodule V of \mathbb{R}_{\max}^n is a *b-complete subsemimodule* of \mathbb{R}_{\max}^n if the supremum of any subset of V bounded from above belongs to V .

Let us also recall that for a b -complete subsemimodule V of \mathbb{R}_{\max}^n the *canonical projection operator* P_V of \mathbb{R}_{\max}^n onto V is defined [8] by

$$P_V(x) := \max\{v \in V \mid v \leq x\}, \quad \forall x \in \mathbb{R}_{\max}^n, \quad (1.8)$$

where \max denotes a supremum which is attained (by some element of V). Then (see [8, 13, 16]) for any $x \in \mathbb{R}_{\max}^n$, $P_V(x)$ is a best approximation of x in V (such a best approximation is not necessarily unique), that is,

$$d(x, P_V(x)) = d(x, V). \quad (1.9)$$

Some of our results are inspired by -and bear some analogy with- those known from the theory of best approximation in normed linear spaces by elements of linear subspaces (see e.g. [21]), reformulated in terms of the “semi-scalar product” (see e.g. [18]). These analogies have led us even to the discovery of some new properties of the canonical projections onto semimodules (see e.g. Theorem 4.3 and Corollaries 4.2, 4.3, 4.5).

The structure of the paper is as follows.

In the preliminary Section 2 we give some notations, concepts and facts that will be used in the sequel, concerning residuation for scalars, vectors and matrices and its connections with the additions $+$ and $+$ ' on $\bar{\mathbb{R}}$, and the Hilbert projective distance d and anti-distance δ on a complete semimodule X , with special emphasis on the particular cases $X = \bar{\mathbb{R}}_{\max}^n$ and $X = \mathbb{R}_{\max}^n$.

In Section 3 we introduce the support, upper support and lower support and the “part” of an element $x \in \bar{\mathbb{R}}_{\max}^n$ and we show that with the aid of these concepts one can reduce the study of best approximation of the elements $x \in \mathbb{R}_{\max}^n$ by the elements of a b -complete subsemimodule V of \mathbb{R}_{\max}^n to the case where $x \in \mathbb{R}^n$ and $V \subset \mathbb{R}^n \cup \{-\infty\}$, where $-\infty$ denotes the vector of \mathbb{R}_{\max}^n with all its entries equal to $-\infty$.

In Section 4, using the known fact [8, 13] that for every b -complete subsemimodule V of \mathbb{R}_{\max}^n and every outside point x there exists a “universal separating half-space” $H = H_{V,x}$, defined with the aid of $P_V(x)$, satisfying $V \subseteq H_{V,x}$ and $x \in \mathbb{R}^n \setminus H_{V,x}$, we show that the problem of best approximation of x by elements of a b -complete subsemimodule V of \mathbb{R}_{\max}^n can be reduced to the problem of best approximation of x by elements of a closed half-space H of \mathbb{R}_{\max}^n . To this end we prove the following properties of H : for each $x \in \mathbb{R}^n \setminus V$ we have $P_V(x) = P_H(x)$ and $d(x, V) = d(x, H)$. As in [8], for more transparency we prove first corresponding results for “complete subsemimodules” of $\bar{\mathbb{R}}_{\max}^n$ and separation by “complete half-spaces” of $\bar{\mathbb{R}}_{\max}^n$, from which we deduce the results on \mathbb{R}_{\max}^n .

In Section 5 we prove for a closed half-space H of \mathbb{R}_{\max}^n and an outside point $x \in \mathbb{R}^n \setminus H$ a formula for the distance $d(x, H)$, and we obtain a formula for the canonical projection $P_H(x)$ of x onto H .

In Section 6 we show that every closed half-space of \mathbb{R}_{\max}^n admits a canonical representation with the aid of coefficients with disjoint supports, and we particularize this result to obtain the canonical form of the universal separating closed half-space of a b -complete subsemimodule V of \mathbb{R}_{\max}^n from

a point $x \notin V$. The latter canonical form shows that when the canonical projection of x onto V is finite, the universal separating closed half-spaces always have “finite apex”.

In Section 7, using the results of Section 6, we give characterizations of the elements of best approximation by arbitrary half-spaces (not necessarily with finite apex) for an element $x \in \mathbb{R}_{\max}^n$. At the end of the section we also give geometric interpretations in simple particular cases.

Finally, in Section 8, as an application of the main distance formula of Section 5, we obtain a new algorithm to solve systems of max-plus linear inequalities $Ax \geq Bx$, where A, B are $p \times n$ matrices. This algorithm uses the technique of cyclic projectors [14]; it may be thought of as a max-plus analogue of the Gauss-Seidel algorithm, and it is shown to be faster than the earlier alternated projection algorithm of [11], although it remains only pseudo-polynomial.

Let us mention that the results on $X = \overline{\mathbb{R}}_{\max}^n$ of this paper can be extended to more general assumptions on a complete semimodule X . To this end, one needs to extend the concept of “opposite” λ^- of (1.3). A rather complete theory of an extension of the “opposite” is developed in [8], but we shall not pursue here that level of generality.

2. NOTATIONS AND PRELIMINARIES

2.1. Residuation. As mentioned above, we denote by \mathbb{R}_{\max} the semiring composed of the set $\mathbb{R} \cup \{-\infty\}$ endowed with the maximization operation as addition $\mu \oplus \nu := \max(\mu, \nu)$, the usual addition as multiplication $\mu \otimes \nu := \mu + \nu$ (also for $\mu = \nu = -\infty$), and the neutral elements $-\infty$ and 0 for addition \oplus and multiplication \otimes respectively. Furthermore, we shall denote by $\overline{\mathbb{R}}_{\max}$ the so-called *complete max-plus algebra*, which is the semiring composed of the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ endowed with the maximization operation as addition, that is,

$$\mu \oplus \nu := \max(\mu, \nu),$$

and with the extension to $\overline{\mathbb{R}}$ of the usual addition $+$ of $\mathbb{R} \cup \{-\infty\}$ as multiplication $\mu \otimes \nu = \mu + \nu$, by the convention

$$a + (+\infty) = (+\infty) + a = \begin{cases} +\infty & \text{if } a \in \mathbb{R} \cup \{+\infty\} \\ -\infty & \text{if } a = -\infty. \end{cases} \quad (2.1)$$

Throughout this paper we shall consider the space \mathbb{R}_{\max}^n (respectively, $\overline{\mathbb{R}}_{\max}^n$) of all n -dimensional column vectors $x = (x_1, \dots, x_n)^T$, where x_1, \dots, x_n belong to \mathbb{R}_{\max} (respectively, $\overline{\mathbb{R}}_{\max}$) and the superscript T denotes the transposition operation, endowed naturally with the pointwise addition (denoted by \oplus) and multiplication by a scalar, that we shall denote by a concatenation on the right. This is a semimodule over \mathbb{R}_{\max} (respectively, $\overline{\mathbb{R}}_{\max}$). We shall denote such column vectors, or equivalently, $n \times 1$ matrices, by the letters x, y, z, u, h, \dots . We shall also consider matrices over \mathbb{R}_{\max} and $\overline{\mathbb{R}}_{\max}$, denoted by capital letters A, B, \dots and employ the usual concatenation notation for

product of matrices, as well as for the multiplication of an element of \mathbb{R}_{\max}^n (or $\overline{\mathbb{R}}_{\max}^n$) by a scalar, that we shall put on the right (as if scalars were one dimensional square matrices). So if $x = (x_1, \dots, x_n)^T \in \mathbb{R}_{\max}^n$ (or $\overline{\mathbb{R}}_{\max}^n$), and $\lambda \in \mathbb{R}_{\max}$ (respectively $\overline{\mathbb{R}}_{\max}$), then $x\lambda$ is the vector $(x_1 + \lambda, \dots, x_n + \lambda)^T$ (the notation $x + \lambda$ is also used in the literature).

As in usual algebra, any max-plus linear operator ϕ from \mathbb{R}_{\max}^n to \mathbb{R}_{\max}^m (respectively $\overline{\mathbb{R}}_{\max}^n$ to $\overline{\mathbb{R}}_{\max}^m$), i.e., satisfying $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$ for all $x, y \in \mathbb{R}_{\max}^n$ (respectively $\overline{\mathbb{R}}_{\max}^n$) and $\phi(x\lambda) = \phi(x)\lambda$ for all $x \in \mathbb{R}_{\max}^n$ ($\overline{\mathbb{R}}_{\max}^n$) and $\lambda \in \mathbb{R}_{\max}$ ($\overline{\mathbb{R}}_{\max}$) can be represented by (and identified to) a $m \times n$ matrix $A = (A_{ij})_{i \in [m], j \in [n]}$ over \mathbb{R}_{\max} (respectively $\overline{\mathbb{R}}_{\max}$), with $\phi(x) = Ax$, that is $\phi(x)_i = \max_{j \in [n]}(A_{ij} + x_j)$ for $i \in [m]$ (see [4]). In particular, when $m = 1$, the dual space $(\mathbb{R}_{\max}^n)^*$ (respectively $(\overline{\mathbb{R}}_{\max}^n)^*$) of all max-plus linear forms over \mathbb{R}_{\max}^n (respectively $\overline{\mathbb{R}}_{\max}^n$), that is, of all max-plus linear functions $(\mathbb{R}_{\max}^n)^* \rightarrow \mathbb{R}_{\max}$ (respectively $(\overline{\mathbb{R}}_{\max}^n)^* \rightarrow \overline{\mathbb{R}}_{\max}$) is isomorphic, and shall be identified, with the space of all n -dimensional row vectors, or equivalently, $1 \times n$ matrices, having their entries in \mathbb{R}_{\max} (respectively, $\overline{\mathbb{R}}_{\max}$), which we shall denote by $a = (a_1, \dots, a_n)$, b, \dots

Spaces of scalars, vectors and matrices over \mathbb{R}_{\max} ($\overline{\mathbb{R}}_{\max}$) are idempotent monoids with respect to addition and their “natural order” for which the supremum operation is equivalent to the addition of the monoid, and that order coincides with the usual partial order. They are b -complete (complete) semimodules over \mathbb{R}_{\max} ($\overline{\mathbb{R}}_{\max}$), in the sense that will be recalled below. This allows one to define the residuation operation $A \setminus B$ for any matrices $A \in \overline{\mathbb{R}}_{\max}^{n \times m}$ and $B \in \overline{\mathbb{R}}_{\max}^{n \times p}$ by

$$A \setminus B := \max\{C \in \overline{\mathbb{R}}_{\max}^{m \times p} \mid AC \leq B\}, \quad (2.2)$$

where the max means that the supremum is attained; in particular, for any scalars $\mu, \nu \in \overline{\mathbb{R}}_{\max}$,

$$\mu \setminus \nu := \max\{\lambda \in \overline{\mathbb{R}}_{\max} \mid \mu \otimes \lambda \leq \nu\}. \quad (2.3)$$

Since semimodules of matrices with entries in \mathbb{R}_{\max} are not complete but only b -complete, the residuation $A \setminus B$ of matrices $A \in \overline{\mathbb{R}}_{\max}^{n \times m}$ and $B \in \overline{\mathbb{R}}_{\max}^{n \times p}$ is not necessarily in $\overline{\mathbb{R}}_{\max}^{m \times p}$; however one can replace the maximum in the definition (2.2) of $A \setminus B$ by the supremum in $\overline{\mathbb{R}}_{\max}^{m \times p}$, as in (1.1).

Let us denote by $\overline{\mathbb{R}}_{\min}$ the so-called *complete min-plus algebra*, which is by definition the semiring composed of the set $\overline{\mathbb{R}}$ endowed with the minimization operation as addition $\mu \oplus' \nu$, that is,

$$\mu \oplus' \nu := \min(\mu, \nu),$$

and with the extension to $\overline{\mathbb{R}}$ of the usual addition $+$ of $\mathbb{R} \cup \{+\infty\}$ as multiplication $\mu \otimes' \nu = \mu +' \nu$, defined by the convention opposite to (2.1), namely:

$$a +' (-\infty) = (-\infty) +' a = \begin{cases} +\infty & \text{if } a = +\infty \\ -\infty & \text{if } a \in \mathbb{R} \cup \{-\infty\}. \end{cases} \quad (2.4)$$

The neutral elements of $\overline{\mathbb{R}}_{\min}$ are necessarily $+\infty$ and 0 for addition $\oplus' = \min$ and multiplication $\otimes' = +'$ respectively.

Remark 2.1. *a)* The above operations $\otimes = +$ and $\otimes' = +'$ are nothing else than the “lower addition” $+$ and “upper addition” $+$ on $\overline{\mathbb{R}}$ respectively, introduced by Moreau (see e.g. [19]) and used extensively in convex analysis. This remark permits to extend the well-known results about $+$ and $+$ on $\overline{\mathbb{R}}$ to the lower and upper product \otimes and \otimes' respectively, on any complete semifield \mathbb{S} , using the known rules for these operations (see e.g. [2]).

b) Here we consider mainly operations of $\overline{\mathbb{R}}_{\max}$, whereas those of $\overline{\mathbb{R}}_{\min}$ are considered as dual ones, hence the notations $+$ and $+'$. Such “dual” notations were already used in the literature, e.g. in [10].

We recall the following well-known rules of computation with $+$ and $+'$ on $\overline{\mathbb{R}}$:

Lemma 2.1. ([19], formulas (2.1) and (2.3)). *For any $\lambda, \mu, \nu \in \overline{\mathbb{R}}$ we have*

$$\begin{aligned} -(\mu +' \nu) &= -\mu + (-\nu), \\ (\lambda +' \mu) +' \nu &= \lambda +' (\mu +' \nu). \end{aligned} \tag{2.5}$$

By (2.5), the semiring $\overline{\mathbb{R}}_{\min}$ can also be defined equivalently as the image of $\overline{\mathbb{R}}_{\max}$ by the “opposite” map $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$, $x \mapsto x^-$ with x^- defined as in (1.3), which means that the opposite map is an isomorphism of complete semirings from $\overline{\mathbb{R}}_{\max}$ to $\overline{\mathbb{R}}_{\min}$.

For the basic rules of computation with residuation of scalars and their extensions to residuation of vectors and matrices see e.g. [4, 8].

Let us give now some new properties of the residuation of scalars that we shall use later.

Proposition 2.1. *For $\mu, \nu \in \overline{\mathbb{R}}_{\max}$, we have*

$$\mu \setminus \nu = \nu +' (-\mu), \tag{2.6}$$

with $+'$ of (2.4).

Proof. By Definition (2.3), we have

$$\mu \setminus \nu := \max\{\lambda \in \overline{\mathbb{R}}_{\max} \mid \mu \otimes \lambda \leq \nu\},$$

that is, in usual notations (with the convention (2.1) for $+$),

$$\mu \setminus \nu = \max\{\lambda \in \overline{\mathbb{R}} \mid \mu + \lambda \leq \nu\}. \tag{2.7}$$

But, by [19], p. 119, Proposition 3(c), for any $\mu, \nu, \lambda \in \overline{\mathbb{R}}$ we have the equivalence

$$\mu + \lambda \leq \nu \Leftrightarrow \lambda \leq \nu +' (-\mu), \tag{2.8}$$

whence, by (2.7) and (2.8), we obtain

$$\mu \setminus \nu = \max\{\lambda \in \overline{\mathbb{R}} \mid \lambda \leq \nu +' (-\mu)\} = \nu +' (-\mu). \quad \square$$

Remark 2.2. For a somewhat similar result see [12, the remark made after Example 3.2].

Corollary 2.1. For $\mu, \nu \in \overline{\mathbb{R}}_{\max}$, we have

$$\mu \setminus \nu \in \mathbb{R} \Leftrightarrow \mu \text{ and } \nu \in \mathbb{R}, \quad (2.9)$$

$$\mu \setminus \nu = +\infty \Leftrightarrow \mu = -\infty \text{ or } \nu = +\infty \text{ (or both).} \quad (2.10)$$

Proof. This follows from Proposition 2.1 and the definition of $+'$, since $-\mu = +\infty$ if and only if $\mu = -\infty$. \square

Remark 2.3. For $\mu, \nu \in \mathbb{R}_{\max}$, we have $\nu < +\infty$, so (2.10) shows that

$$\mu \setminus \nu = +\infty \Leftrightarrow \mu = -\infty.$$

Hence, for $x, y \in \mathbb{R}_{\max}^n$, we have the following equivalence

$$x \setminus y = +\infty \Leftrightarrow x = -\infty \quad (\text{that is, } x_i = -\infty, \forall i \in [n]). \quad (2.11)$$

Since $\mu \setminus \nu$ is an element of $\overline{\mathbb{R}}$, we get by taking the complementaries of the equivalences (2.9) and (2.10):

Corollary 2.2. We have

$$\begin{aligned} \mu \setminus \nu = -\infty &\Leftrightarrow (\mu \text{ or } \nu \notin \mathbb{R}) \text{ and } \mu > -\infty, \nu < +\infty \\ &\Leftrightarrow (\mu = +\infty \text{ and } \nu < +\infty) \text{ or } (\mu > -\infty \text{ and } \nu = -\infty). \end{aligned}$$

By the above, we can summarize all possible values of $\mu \setminus \nu$ in the following table:

		ν	$-\infty$	real	$+\infty$
		μ	$-\infty$	$+\infty$	$+\infty$
μ	$-\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
real	$-\infty$	$-\infty$	real	$+\infty$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$

Remark 2.4. a) Definition (2.2) gives that for any vectors $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_n)^T \in \overline{\mathbb{R}}_{\max}^n$, we have

$$\begin{aligned} x \setminus y &= \max\{\lambda \in \overline{\mathbb{R}}_{\max} \mid x\lambda \leq y\} \\ &= \max\{\lambda \in \overline{\mathbb{R}}_{\max} \mid x_i \otimes \lambda \leq y_i (i \in [n])\} \\ &= \wedge_{i \in [n]} x_i \setminus y_i, \end{aligned} \quad (2.12)$$

where $[n] = \{1, \dots, n\}$ and \wedge denotes the infimum operation. Hence, using also (2.6),

$$\begin{aligned} x \setminus y &= \max\{\lambda \in \overline{\mathbb{R}} \mid x_i + \lambda \leq y_i (i \in [n])\} \\ &= \min_{i \in [n]} (y_i +' (-x_i)). \end{aligned} \quad (2.13)$$

b) By (2.13) and (2.4), for any $x \in \overline{\mathbb{R}}_{\max}^n$ we have

$$x \setminus x = \wedge_{i \in [n]} x_i \setminus x_i = \min_{i \in [n]} (x_i +' (-x_i)) = \begin{cases} +\infty & \text{if } x \in \{-\infty, +\infty\}^n \\ 0 & \text{if } x \notin \{-\infty, +\infty\}^n. \end{cases} \quad (2.14)$$

c) By (2.12) we have the following equivalence:

$$\lambda \leq x \setminus y \Leftrightarrow x\lambda \leq y, \quad (2.15)$$

for all $\lambda \in \overline{\mathbb{R}}_{\max}$ and $x, y \in \overline{\mathbb{R}}_{\max}^n$ and for all $\lambda \in \mathbb{R}_{\max}$ and $x, y \in \mathbb{R}_{\max}^n$.

2.2. Hilbert projective distance. For a complete semimodule X over a complete idempotent semiring \mathbb{S} and for any $x, y \in X$, let us set

$$\delta(x, y) := (x \setminus y) \otimes (y \setminus x), \quad (2.16)$$

where \otimes denotes the multiplication of \mathbb{S} . The last part of the following result of [8] shows that when \mathbb{S} is commutative, the mapping $\delta : X \times X \rightarrow \mathbb{S}$ satisfies an inequality opposite to the triangular inequality for a distance, and thus $\delta(x, y)$ may be called an “anti-distance”; by abuse of language, we shall also keep this term in the non-commutative case, even when δ is not symmetrical. Recall that, since \mathbb{S} is complete, the partial order relation defined by (1.7) determines an infimum operation, denoted by \wedge , see [8]. In what follows, we denote by $\mathbf{1}$ the unit element of \mathbb{S} .

Proposition 2.2. [8, Theorem 17] *Let X be a complete semimodule over a complete idempotent commutative semiring \mathbb{S} . Then, for any $x, y, z \in X$, we have*

$$\delta(x, y) \leq (x \setminus x) \wedge (y \setminus y), \quad (2.17)$$

$$\delta(x, y) = \mathbf{1} \Rightarrow y = x\lambda, \text{ for some } \lambda \in \mathbb{S}, \quad (2.18)$$

$$\delta(x, z) \geq \delta(x, y) \otimes \delta(y, z). \quad (2.19)$$

Following [7, 8, 13], we define the Hilbert projective distance d on $\overline{\mathbb{R}}_{\max}^n$ by

$$d(x, y) := \delta(x, y)^-, \quad (2.20)$$

with δ of (2.16), that is, by the same expression (1.2) as on \mathbb{R}_{\max}^n , where the superscript $-$ is defined on $\overline{\mathbb{R}}$ by (1.3). For brevity, in the sequel by “distance” we shall always mean the Hilbert projective distance.

Corollary 2.3. *For $x, y, z \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$, we have*

$$d(x, y) \geq 0, \quad (2.21)$$

$$d(x, y) = 0 \Leftrightarrow x = y\lambda, \text{ for some } \lambda \in \mathbb{R}, \quad (2.22)$$

$$d(x, z) \leq d(x, y) + d(y, z). \quad (2.23)$$

More generally, for all $x, y, z \in \overline{\mathbb{R}}_{\max}^n$, we have

$$d(x, z) \leq d(x, y) +' d(y, z), \quad (2.24)$$

and the implication

$$d(x, y) = 0 \Rightarrow x = y\lambda, \text{ for some } \lambda \in \mathbb{R}. \quad (2.25)$$

(Recall that in the present setting, $y\lambda$ is now the vector with entries $y_i + \lambda$, for $i \in [n]$.)

This result means that if one avoids vectors with only infinite entries, then d satisfies all properties of a projective distance, except that it may

take infinite values. The term “projective” comes from (2.22). This result was given without proof in [7, p. 6]. For the sake of completeness, we give here a proof, using Proposition 2.2.

Proof. (2.21): If $x, y \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$, then by (2.14) we have $x \setminus x = y \setminus y = 0$, and hence by (2.17) we obtain $\delta(x, y) \leq 0 \wedge 0 = 0$, so $d(x, y) = \delta(x, y)^- \geq 0$.

(2.25), (2.22): If $x, y \in \overline{\mathbb{R}}_{\max}^n$ and $d(x, y) = 0$, then $\delta(x, y) = d(x, y)^- = 0^- = 0 = \mathbf{1}$, and hence by (2.18) there exists $\lambda \in \overline{\mathbb{R}}_{\max}$ such that $y = x\lambda$. Thus, we have the implication \Rightarrow in (2.25) and (2.22). Conversely, if $y = x\lambda$, where $x, y \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$ and $\lambda \in \mathbb{R}$, then $x \setminus y = x \setminus (x\lambda) = (x + \lambda) - x = \lambda$ and $y \setminus x = (x\lambda) \setminus x = x - (x + \lambda) = -\lambda = \lambda^-$, whence

$$d(x, y) = d(x, x\lambda) = ((x \setminus (x\lambda)) \otimes ((x\lambda) \setminus x))^+ = \lambda^- \otimes \lambda = 0.$$

(2.24), (2.23): Taking the opposite of (2.19) and using (2.5), we get

$$\begin{aligned} d(x, z) &= \delta(x, z)^- \leq (\delta(x, y) \otimes \delta(y, z))^+ \\ &= \delta(y, z)^- \otimes' \delta(x, y)^- = d(y, z) +' d(x, y), \end{aligned}$$

that is, (2.24). Finally, if $x, y, z \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$, then by (2.21) all the quantities in (2.24) are nonnegative, hence (2.24) reduces to (2.23). ■

Remark 2.5. *a)* (2.21) does not hold for $x = y \in \{-\infty, +\infty\}^n$. Indeed, then $x \setminus x = +\infty$ (by (2.14)), whence $\delta(x, x) = +\infty$, so

$$d(x, x) = \delta(x, x)^- = -\infty, \quad \forall x \in \{-\infty, +\infty\}^n. \quad (2.26)$$

Consequently, if V is a subsemimodule of $\overline{\mathbb{R}}_{\max}^n$ (so $-\infty \in V$), then by (2.26) we have

$$d(-\infty, V) = \inf_{v \in V} d(-\infty, v) = -\infty, \quad (2.27)$$

which shows that the best approximation of $x = -\infty$ by V is trivial.

b) The implication \Leftarrow of (2.22) does not hold if $x \in \{-\infty, +\infty\}^n$, since then the right hand side of (2.22) implies that $x = y$, but then by *a*) above, $d(x, y) = -\infty$, so that the left hand side of (2.22) does not hold.

c) In general one cannot replace $+'$ by $+$ in (2.24). Indeed, for example if $x = y \in \{-\infty, +\infty\}^n$ and $z \in \mathbb{R}^n$, then $d(x, z) = +\infty$ and $d(x, y) = -\infty$ (by (2.26)), $d(y, z) = +\infty$, so $d(x, z) \not\leq d(x, y) + d(y, z)$.

d) In the sequel in the proofs of some statements about d we shall rather work with δ instead of d in order to use only $+$ instead of both $+'$ and $+$, and then only in the final step of the proof we shall pass to the conclusion for $d(x, y) = \delta(x, y)^-$.

In [8] the results are presented in the case of complete subsemimodules of a complete semimodule. Here we shall consider mainly the projection onto, and the best approximation by elements, of a *b*-complete subsemimodule of $\overline{\mathbb{R}}_{\max}^n$, but as far as possible, we shall write the results also for complete subsemimodules of $\overline{\mathbb{R}}_{\max}^n$. Recall that a subsemimodule V of $\overline{\mathbb{R}}_{\max}^n$ is called *complete* if the supremum (or infinite sum) of any subset of V belongs to V

and the scalar multiplication $\otimes = +$ distributes over all infinite sums. If V is a subset (and in particular a subsemimodule) of $\overline{\mathbb{R}}_{\max}^n$, and $x \in \overline{\mathbb{R}}_{\max}^n$, one simply defines the *distance* $d(x, V)$ of x to V and the *best approximation* of x by an element v_0 of V as in the case of \mathbb{R}_{\max}^n , that is by (1.5) and (1.6). For a complete subsemimodule V of $\overline{\mathbb{R}}_{\max}^n$, the *canonical projection operator* P_V of $\overline{\mathbb{R}}_{\max}^n$ onto V is also defined [8] by (1.8) for all $x \in \overline{\mathbb{R}}_{\max}^n$.

2.3. An equivalent reformulation of the inequality $Ax \geq Bx$. For a later application of our main distance formula to the solution of the system of inequalities

$$Ax \geq Bx, \quad (2.28)$$

where $A, B : \mathbb{R}_{\max}^n \rightarrow \mathbb{R}_{\max}^p$ are $p \times n$ matrices with entries in \mathbb{R}_{\max} , we give here an equivalent reformulation of (2.28). We denote by A_i and B_i the i th rows of A and B , respectively. We recall that for B as above, which in addition satisfies the assumption

$$\text{for all } j \in [n] \text{ there exists } i \in [p] \text{ such that } B_{ij} \neq -\infty, \quad (2.29)$$

one defines (see e.g. [4], [1] and the references therein) the *residuated operator* $B^\#$ from \mathbb{R}_{\max}^p to \mathbb{R}_{\max}^n by

$$(B^\# y)_j = \inf_i (-B_{ij} +' y_i). \quad (2.30)$$

We shall assume that the matrix B satisfies assumption (2.29).

The term “residuated” refers to the well-known equivalence of the inequalities

$$Bx \leq y \Leftrightarrow x \leq B^\# y. \quad (2.31)$$

We recall the easy proof of this equivalence: We have

$$\begin{aligned} Bx \leq y &\Leftrightarrow (Bx)_i \leq y_i, \forall 1 \leq i \leq p \Leftrightarrow \sup_j (B_{ij} + x_j) \leq y_i, \forall 1 \leq i \leq p \\ &\Leftrightarrow B_{ij} + x_j \leq y_i, \forall 1 \leq i \leq p, \forall 1 \leq j \leq n \\ &\Leftrightarrow x_j \leq -B_{ij} +' y_i, \forall 1 \leq i \leq p, \forall 1 \leq j \leq n \\ &\Leftrightarrow x_j \leq \inf_i (-B_{ij} +' y_i) = (B^\# y)_j, \forall 1 \leq j \leq n \Leftrightarrow x \leq B^\# y. \end{aligned}$$

Note that (2.30) is a particular case of residuation operators for matrices in the sense (2.2), since regarding $y \in \mathbb{R}_{\max}^p$ as a $p \times 1$ matrix we have

$$B^\# y = B \setminus y;$$

indeed, using (2.31), we obtain

$$B^\# y = \max\{x \in \mathbb{R}_{\max}^p | x \leq B^\# y\} = \max\{x \in \mathbb{R}_{\max}^p | Bx \leq y\} = B \setminus y.$$

Applying (2.31) to $y = Ax$, we get

$$Bx \leq Ax \Leftrightarrow x \leq B^\# Ax. \quad (2.32)$$

Finally, since the right hand side of (2.32) can be written in the form of the equality $x = B^\# Ax \wedge x$, we obtain the equivalence

$$Bx \leq Ax \Leftrightarrow x = B^\# Ax \wedge x, \quad (2.33)$$

which we shall use later on.

3. ON THE DISTANCE TO A SUBSEMIMODULE OF \mathbb{R}_{\max}^n

Next we shall give some properties of the distance d of (1.2) and we shall show that one may reduce the study of the best approximation of elements $x \in \mathbb{R}_{\max}^n$ by the elements of a b -complete subsemimodule V of \mathbb{R}_{\max}^n to the case where $x \in \mathbb{R}^n$ and $V \subset \mathbb{R}^n \cup \{-\infty\}$.

Definition 3.1. For an element $x = (x_1, \dots, x_n)^T$ of $\overline{\mathbb{R}}_{\max}^n$, we define the *support* $\text{Supp } x$, *lower support* $\text{Lsupp } x$ and *upper support* $\text{Usupp } x$ of x by:

$$\begin{aligned}\text{Supp } x &:= \{i \in [n] \mid x_i \in \mathbb{R}\}, \\ \text{Lsupp } x &:= \{i \in [n] \mid x_i < +\infty\}, \\ \text{Usupp } x &:= \{i \in [n] \mid x_i > -\infty\}.\end{aligned}$$

We have trivially

$$\text{Supp } x = \text{Lsupp } x \cap \text{Usupp } x. \quad (3.1)$$

Moreover, when $x \in \mathbb{R}_{\max}^n$, we have

$$\text{Lsupp } x = [n].$$

Lemma 3.1. For any $x, y \in \overline{\mathbb{R}}_{\max}^n$ the following statements are equivalent:

- 1°. $x \setminus y > -\infty$.
- 2°. There exists $\lambda \in \mathbb{R}$ such that $x\lambda \leq y$, that is, $x_i + \lambda \leq y_i$ for all $i \in [n]$.
- 3°. We have

$$\text{Usupp } x \subset \text{Usupp } y, \quad \text{Lsupp } x \supset \text{Lsupp } y. \quad (3.2)$$

Proof. $1^\circ \Rightarrow 2^\circ$. Let $x, y \in \overline{\mathbb{R}}_{\max}^n$ be such that $x \setminus y > -\infty$. Then there exists $\lambda \in \mathbb{R}$ such that $\lambda \leq x \setminus y$ and hence by (2.15), $x\lambda \leq y$.

$2^\circ \Rightarrow 3^\circ$. Let $x, y \in \overline{\mathbb{R}}_{\max}^n$ and $\lambda \in \mathbb{R}$ be such that $x\lambda \leq y$, that is, $x_i + \lambda \leq y_i$ for all $i \in [n]$. It follows that if $x_i > -\infty$ then $y_i > -\infty$, which shows the inclusion $\text{Usupp } x \subset \text{Usupp } y$. Similarly, if $y_i < +\infty$ then $x_i < +\infty$, which shows the inclusion $\text{Lsupp } y \subset \text{Lsupp } x$.

$3^\circ \Rightarrow 1^\circ$. Assume that $x, y \in \overline{\mathbb{R}}_{\max}^n$ satisfy (3.2). Since $x \setminus y = \min_{i \in [n]} x_i \setminus y_i$, we get that $x \setminus y > -\infty$ if and only $x_i \setminus y_i > -\infty$ for all $i \in [n]$. Now, if $y_i = -\infty$ then $i \in [n] \setminus \text{Usupp } y$ so by (3.2), $i \in [n] \setminus \text{Usupp } x$, that is $x_i = -\infty$, whence by (2.10), $x_i \setminus y_i = +\infty > -\infty$. Similarly, if $x_i = +\infty$ then $i \in [n] \setminus \text{Lsupp } x$ so by (3.2), $i \in [n] \setminus \text{Lsupp } y$, that is $y_i = +\infty$, whence by (2.10), $x_i \setminus y_i = +\infty > -\infty$. Otherwise, $y_i > -\infty$ and $x_i < +\infty$, so there exist λ and $\mu \in \mathbb{R}$ such that $y_i \geq \lambda$ and $x_i \leq \mu$, whence $x_i \setminus y_i \geq \mu \setminus \lambda \in \mathbb{R}$. ■

As for convex sets without lines in linear spaces (see e.g. [5, 22]), we define the *part* of an element of $\overline{\mathbb{R}}_{\max}^n$ as follows:

Definition 3.2. The part $[[x]]$ of $x = (x_1, \dots, x_n)^T \in \overline{\mathbb{R}}_{\max}^n$ is the equivalence class of x for the equivalence relation (of comparability)

$$\begin{aligned} x \sim y &\text{ if there exist } \lambda, \mu \in \mathbb{R} \text{ such that } x\lambda \leq y \leq x\mu, \\ &\text{that is, } x_i + \lambda \leq y_i \leq x_i + \mu \quad \forall i \in [n]. \end{aligned}$$

Applying Lemma 3.1 symmetrically on x and y , we deduce

Lemma 3.2. *The following statements are equivalent for $x, y \in \overline{\mathbb{R}}_{\max}^n$:*

- 1°. $d(x, y) < +\infty$.
- 2°. x, y are in the same part.
- 3°. We have

$$\text{Usupp } x = \text{Usupp } y, \quad \text{Lsupp } x = \text{Lsupp } y. \quad (3.3)$$

- 4°. We have

$$\text{Supp } x = \text{Supp } y, \quad \sigma_{-\infty}(x) = \sigma_{-\infty}(y) \text{ and } \sigma_{+\infty}(x) = \sigma_{+\infty}(y),$$

where for any $x \in \overline{\mathbb{R}}_{\max}^n$ and $\lambda = \pm\infty$ we denote

$$\sigma_\lambda(x) := \{i \in [n] \mid x_i = \lambda\}.$$

Proof. $1^\circ \Rightarrow 2^\circ$. If $d(x, y) < +\infty$, then $(x \setminus y) \otimes (y \setminus x) = \delta(x, y) = d(x, y)^- > -\infty$, which implies that both $x \setminus y$ and $y \setminus x$ are $> -\infty$, since $-\infty \otimes \mu = \mu \otimes -\infty = -\infty$ for all $\mu \in \overline{\mathbb{R}}_{\max}$. Hence by the implication $1^\circ \Rightarrow 2^\circ$ of Lemma 3.1, we get that there exist $\lambda, \mu \in \mathbb{R}$ such that $x\lambda \leq y$ and $y\mu \leq x$. Since $\mu \in \mathbb{R}$ is invertible, it follows that $x\lambda \leq y \leq x\mu^{-1}$, hence x and y are in the same part.

$2^\circ \Rightarrow 3^\circ$. Assume 2° , so there exist $\lambda, \mu \in \mathbb{R}$ such that $x\lambda \leq y \leq x\mu$. By the implication $2^\circ \Rightarrow 3^\circ$ of Lemma 3.1, we get the inclusions (3.2). But, since $\mu, \lambda \in \mathbb{R}$ are invertible, we have $y\mu^{-1} \leq x \leq y\lambda^{-1}$, so that 2° also holds for the pair (y, x) . Consequently, we get the opposite inclusions to (3.2), whence the equalities (3.3).

$3^\circ \Rightarrow 1^\circ$. Assume now the equalities (3.3). By the implication $3^\circ \Rightarrow 1^\circ$ of Lemma 3.1 applied to the two pairs (x, y) and (y, x) , we get that $x \setminus y > -\infty$ and $y \setminus x > -\infty$. This implies that $\delta(x, y) = (x \setminus y) \otimes (y \setminus x) > -\infty$, whence $d(x, y) = \delta(x, y)^- < +\infty$.

$3^\circ \Rightarrow 4^\circ$. This follows from the fact that for all $x \in \overline{\mathbb{R}}_{\max}^n$ we have (3.1) and

$$\sigma_{-\infty}(x) = \text{Lsupp } x \setminus \text{Usupp } x, \quad \sigma_{+\infty}(x) = \text{Usupp } x \setminus \text{Lsupp } x.$$

$4^\circ \Rightarrow 3^\circ$. Similarly this follows from the fact that for all $x \in \overline{\mathbb{R}}_{\max}^n$ we have

$$\text{Lsupp } x = \text{Supp } x \cup \sigma_{-\infty}(x), \quad \text{Usupp } x = \text{Supp } x \cup \sigma_{+\infty}(x). \quad \blacksquare$$

Remark 3.1. a) The equivalence $2^\circ \Leftrightarrow 1^\circ$ of Lemma 3.2 can be expressed in the form of the following useful formula for the part of x :

$$[[x]] = \{y \in \overline{\mathbb{R}}_{\max}^n \mid d(x, y) < +\infty\} \quad \forall x \in \overline{\mathbb{R}}_{\max}^n. \quad (3.4)$$

Hence, in particular, for any subset V of $\overline{\mathbb{R}}_{\max}^n$ we have

$$V \cap [[x]] = \{v \in V \mid d(x, v) < +\infty\} \quad \forall x \in \overline{\mathbb{R}}_{\max}^n. \quad (3.5)$$

b) Similarly, the equivalences $2^\circ \Leftrightarrow 3^\circ$ and $2^\circ \Leftrightarrow 4^\circ$ of Lemma 3.2 can be expressed as formulas for the part of x , namely:

$$\begin{aligned} [[x]] &= \{y \in \overline{\mathbb{R}}_{\max}^n \mid \text{Usupp } y = \text{Usupp } x, \text{ Lsupp } y = \text{Lsupp } x\}, \\ [[x]] &= \{y \in \overline{\mathbb{R}}_{\max}^n \mid \text{Supp } y = \text{Supp } x, \sigma_\lambda(y) = \sigma_\lambda(x) (\lambda = \pm\infty)\}. \end{aligned}$$

Corollary 3.1. *For each $x \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$ we have $[[x]] \subset \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$ and d is a projective distance on $[[x]]$.*

Proof. Let $x \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty, +\infty\}^n$ and $y \in [[x]]$. If $y \in \{-\infty, +\infty\}^n$, then $\text{Supp } y = \{i \in [n] \mid y_i \in \mathbb{R}\} = \emptyset$, whence by $y \in [[x]]$ and the implication $2^\circ \Rightarrow 4^\circ$ of Lemma 3.2, we obtain $\text{Supp } x = \emptyset$, so $x \in \{-\infty, +\infty\}^n$, which contradicts our assumption. Therefore we must have $y \notin \{-\infty, +\infty\}^n$, which proves the first assertion of the corollary. Finally, the second assertion of the corollary holds by (3.4). ■

Corollary 3.2. *For any $x \in \{-\infty, +\infty\}^n$ the part of x is reduced to the singleton $\{x\}$, that is:*

$$[[x]] = \{x\} \quad \forall x \in \{-\infty, +\infty\}^n, \quad (3.6)$$

and hence, in particular, $\{-\infty\}$ is a part of $\overline{\mathbb{R}}_{\max}^n$ or \mathbb{R}_{\max}^n . Also, on $[[x]]$, d is identically equal to $-\infty$.

Proof. For any $x \in \{-\infty, +\infty\}^n$ we have $\text{Supp } x = \emptyset$ and all the entries of x are determined by $\sigma_{-\infty}(x)$:

$$x_i = \begin{cases} -\infty & \text{for } i \in \sigma_{-\infty}(x) \\ +\infty & \text{for } i \notin \sigma_{-\infty}(x). \end{cases}$$

Then, the equivalence $2^\circ \Leftrightarrow 4^\circ$ of Lemma 3.2 implies that $y \in [[x]]$ if and only if $y = x$, which shows that $[[x]] = \{x\}$. Hence in particular, $[[\{-\infty\}]] = \{-\infty\}$, so $\{-\infty\}$ is a part of $\overline{\mathbb{R}}_{\max}^n$ or \mathbb{R}_{\max}^n . Also, by (2.26), d is identically equal to $-\infty$ on $[[x]]$, for any $x \in \{-\infty, +\infty\}^n$. ■

The main application to best approximation is the following:

Theorem 3.1. *If V is a subset of \mathbb{R}_{\max}^n (or $\overline{\mathbb{R}}_{\max}^n$) and $x \in \mathbb{R}_{\max}^n$ (or $\overline{\mathbb{R}}_{\max}^n$), then $d(x, V) < +\infty$ if and only if V intersects the part $[[x]]$ of x (i.e., $V \cap [[x]] \neq \emptyset$), and in that case*

$$d(x, V) < d(x, v) = +\infty, \quad \forall v \in V \setminus [[x]], \quad (3.7)$$

so any best approximation of x in V is necessarily in $[[x]]$, and

$$d(x, V) = d(x, V \cap [[x]]). \quad (3.8)$$

Proof. Assume that $d(x, V) < +\infty$. Then $\inf_{v \in V} d(x, v) < +\infty$, so there exists $v \in V$ such that $d(x, v) < +\infty$. By (3.4), we must have $v \in [[x]]$, so $V \cap [[x]] \neq \emptyset$.

Conversely, assume that $V \cap [[x]] \neq \emptyset$, say $v \in V \cap [[x]]$. Then by $v \in [[x]]$ and (3.4), we have $d(x, v) < +\infty$, so by $v \in V$ we obtain $d(x, V) \leq d(x, v) < +\infty$. This proves the equivalence $d(x, V) < +\infty \Leftrightarrow V \cap [[x]] \neq \emptyset$. Moreover, by (3.4) we have $d(x, v) = +\infty$ when $v \notin [[x]]$, which shows formula (3.7), whence also $d(x, V \setminus [[x]]) = \inf_{v \in V \setminus [[x]]} d(x, v) = +\infty$, and any best approximation of x in V is necessarily in $[[x]]$. Since V is the disjoint union $V = (V \cap [[x]]) \cup (V \setminus [[x]])$, we obtain

$$d(x, V) = \min\{d(x, V \cap [[x]]), d(x, V \setminus [[x]])\} = d(x, V \cap [[x]]).$$

■

The first part of Theorem 3.1 can be also expressed in the following useful form:

Corollary 3.3. *For any subsemimodule V of $\overline{\mathbb{R}}_{\max}^n$ we have*

$$\{x \in \overline{\mathbb{R}}_{\max}^n \mid d(x, V) < +\infty\} = \{x \in \overline{\mathbb{R}}_{\max}^n \mid V \cap [[x]] \neq \emptyset\}.$$

In the sequel we shall give some results in $\overline{\mathbb{R}}_{\max}^n$.

Corollary 3.4 (in $\overline{\mathbb{R}}_{\max}^n$). a) *For $x \in \overline{\mathbb{R}}_{\max}^n$, we have $[[x]] \subset \overline{\mathbb{R}}_{\max}^n$ and*

$$[[x]] = \{y \in \overline{\mathbb{R}}_{\max}^n \mid \text{Supp } y = \text{Supp } x\}. \quad (3.9)$$

b) *For $x \in \overline{\mathbb{R}}_{\max}^n$, we have*

$$d(x, -\infty) = \begin{cases} +\infty & \text{if } x > -\infty \\ -\infty & \text{if } x = -\infty. \end{cases} \quad (3.10)$$

Proof. a) By the implication $2^\circ \Rightarrow 3^\circ$ of Lemma 3.2 and the obvious equivalence

$$x \in \overline{\mathbb{R}}_{\max}^n \Leftrightarrow (x \in \overline{\mathbb{R}}_{\max}^n, \text{Lsupp } x = [n]), \quad (3.11)$$

we get for any $x \in \overline{\mathbb{R}}_{\max}^n$ that $y \in [[x]]$ implies $\text{Lsupp } y = \text{Lsupp } x = [n]$, whence $y \in \overline{\mathbb{R}}_{\max}^n$; thus $[[x]] \subset \overline{\mathbb{R}}_{\max}^n$ for any element $x \in \overline{\mathbb{R}}_{\max}^n$. Moreover, if $x \in \overline{\mathbb{R}}_{\max}^n$, then $\text{Usupp } x = \text{Supp } x$ and hence, by (3.11) and the equivalence $2^\circ \Leftrightarrow 3^\circ$ of Lemma 3.2 we have $y \in [[x]]$ if and only if $\text{Supp } y = \text{Supp } x$.

b) If $x > -\infty$ then $\text{Supp } x \neq \emptyset$, and since $\text{Supp } (-\infty) = \emptyset$, x and $-\infty$ are in different parts. Hence, by (3.4), $d(x, -\infty) = +\infty$. On the other hand, by (2.26) we have $d(-\infty, -\infty) = -\infty$. ■

Remark 3.2. a) In particular, \mathbb{R}^n is a part of $\overline{\mathbb{R}}_{\max}^n$; indeed, the points in \mathbb{R}^n are exactly those that have support equal to the set $[n]$, and hence, by (3.9), all points in \mathbb{R}^n are in the same part as one of them, say 0.

b) When $x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T \in \overline{\mathbb{R}}_{\max}^n \setminus \{-\infty\}$ have the same support $I \subset [n]$, we have

$$d(x, y) = \max_{i \in I} (x_i - y_i) - \min_{j \in I} (x_j - y_j). \quad (3.12)$$

Indeed, if $k \notin I := \text{Supp } x = \text{Supp } y$, then $x_k = y_k = -\infty$, so $x_k \setminus y_k = +\infty$ (see (2.11)), and hence

$$\begin{aligned} x \setminus y &= \min_{i \in [n]} (x_i \setminus y_i) = \min \left\{ \min_{i \in I} (x_i \setminus y_i), \min_{k \in [n] \setminus I} (x_k \setminus y_k) \right\} \\ &= \min_{i \in I} (x_i \setminus y_i) = \min_{i \in I} (y_i - x_i), \end{aligned}$$

where the terms in the latter expression are all finite. Similarly, $y \setminus x = \min_{j \in I} (x_j - y_j)$. Consequently, we obtain

$$\begin{aligned} d(x, y) &= ((x \setminus y) \otimes (y \setminus x))^- \\ &= -(\min_{i \in I} (y_i - x_i) + \min_{j \in I} (x_j - y_j)) \\ &= \max_{i \in I} (x_i - y_i) - \min_{j \in I} (x_j - y_j), \end{aligned}$$

that is, (3.12).

c) Combining Remark 3.1 and Corollary 3.4a), it follows that in \mathbb{R}_{\max}^n we have the equivalence

$$d(x, y) < +\infty \Leftrightarrow \text{Supp } y = \text{Supp } x. \quad (3.13)$$

Note that this also follows from Lemma 3.2, equivalence $1^\circ \Leftrightarrow 4^\circ$.

Remark 3.3. When $x, y \in \mathbb{R}_{\max}^n \setminus \{-\infty\}$, the Hilbert projective distance $d(x, y)$ can be characterized by

$$d(x, y) = \inf \left\{ \frac{\mu}{\lambda} \mid \lambda \in \mathbb{R}, \mu \in \mathbb{R} \quad y\lambda \leq x \leq y\mu \right\}. \quad (3.14)$$

To show this, we shall assume that x and y have the same support (otherwise, the set in (3.14) is empty, so its infimum, $+\infty$, trivially coincides with $d(x, y) = +\infty$). Then, the maximal $\lambda \in \mathbb{R}$ such that $y\lambda \leq x$ is $\min_{j \in I} (x_j - y_j)$. Similarly, the term $\max_{i \in I} (x_i - y_i)$ coincides with the minimal $\mu \in \mathbb{R}$ such that $x \leq y\mu$. Therefore, (3.14) coincides with the expression of $d(x, y)$ in (3.12).

Using formula (1.9), we get as a corollary of Theorem 3.1:

Corollary 3.5. a) If V is a b-complete subsemimodule of \mathbb{R}_{\max}^n , and $d(x, V) < +\infty$, then $P_V(x) \in V \cap [[x]]$.

b) Consequently, if V is a b-complete subsemimodule of \mathbb{R}_{\max}^n , and $V \cap [[x]] \neq \emptyset$, then $P_V(x) \in V \cap [[x]]$.

Proof. a) We have $P_V(x) \in V$ by the definition (1.8) of $P_V(x)$. Furthermore, by (1.9) and our assumption we have $d(x, P_V(x)) = d(x, V) < +\infty$, and hence by (3.7) we obtain $P_V(x) \in [[x]]$.

b) This follows from (3.5) and part a). ■

Proposition 3.1. If V is a (b-complete) subsemimodule of \mathbb{R}_{\max}^n , then the set

$$V^{(x)} := (V \cap [[x]]) \cup \{-\infty\} \quad (3.15)$$

is the smallest (*b*-complete) subsemimodule of \mathbb{R}_{\max}^n containing $V \cap [[x]]$ and we have

$$d(x, V) = d(x, V^{(x)}). \quad (3.16)$$

Proof. Assume that V is a *b*-complete subsemimodule of \mathbb{R}_{\max}^n and $x \in \mathbb{R}_{\max}^n$. Since any subsemimodule of \mathbb{R}_{\max}^n contains necessarily $-\infty$, any subsemimodule of \mathbb{R}_{\max}^n containing $V \cap [[x]]$ necessarily contains $V^{(x)} = (V \cap [[x]]) \cup \{-\infty\}$. Moreover, since $-\infty \in V$, by (3.15) we have $(V \cap [[x]]) \subset V^{(x)} \subset V$, whence, by (3.8) and (3.10), we obtain (3.16).

Now let us prove that $V^{(x)}$ is a *b*-complete subsemimodule of \mathbb{R}_{\max}^n . It is easy to see that $[[x]] \cup \{-\infty\}$ is a subsemimodule of \mathbb{R}_{\max}^n , since for all $y, z \in \mathbb{R}_{\max}^n$ and $\lambda \in \mathbb{R}_{\max}$, we have $\text{Supp}(y \oplus z) = \text{Supp } y \cup \text{Supp } z$ and $\text{Supp } y\lambda = \text{Supp } y$ if $\lambda \neq -\infty$ and $\text{Supp } y\lambda = \emptyset$ otherwise. Hence, since V is a subsemimodule of \mathbb{R}_{\max}^n , so is also the set

$$V^{(x)} = (V \cap [[x]]) \cup (V \cap \{-\infty\}) = V \cap ([[x]] \cup \{-\infty\}). \quad (3.17)$$

Let us show that $V^{(x)}$ is *b*-complete. Let M be a subset of $V^{(x)}$ bounded from above by an element of $V^{(x)}$. Since $V^{(x)} \subset V$, then M is also a subset of V bounded from above by an element of V , and since V is *b*-complete, then M admits a supremum in V . Let us denote it by m and show that it belongs to $V^{(x)}$. If $m = -\infty$, then $m \in V^{(x)}$ and we are done. Otherwise, there exists $y \in M \setminus \{-\infty\} \subset [[x]]$. Since $m \geq y$, we get that $\text{Supp } m \supset \text{Supp } y$ (by the implication $2^\circ \Rightarrow 3^\circ$ of Lemma 3.1) and since $\text{Supp } y = \text{Supp } x$ for all $y \in [[x]]$ (by Corollary 3.4, a)), we obtain $\text{Supp } m \supset \text{Supp } x$. Conversely, if $i \notin \text{Supp } x$, then $y_i = -\infty$ for all $y \in [[x]] \cup \{-\infty\}$, hence for all $y \in M \subset V^{(x)} \subset [[x]] \cup \{-\infty\}$, which implies that $m_i = \sup\{y_i \mid y \in M\} = -\infty$. This shows that $\text{Supp } m \subset \text{Supp } x$, hence the equality, which is equivalent to the property that $m \in [[x]]$ (again by Corollary 3.4, a)). This implies that $m \in [[x]] \cap V \subset V^{(x)}$, and shows that $V^{(x)}$ is a *b*-complete subsemimodule of \mathbb{R}_{\max}^n . ■

Now we shall show that one can reduce the study of the best approximation of elements $x \in \mathbb{R}_{\max}^n$ by the elements of a *b*-complete subsemimodule V of \mathbb{R}_{\max}^n to the case where

$$x \in \mathbb{R}^{n'}, V \subset \mathbb{R}^{n'} \cup \{-\infty\}, \quad (3.18)$$

with a suitable $n' \leq n$ depending on x . To this end, for any $I \subset [n]$ and $x \in \mathbb{R}_{\max}^n$, let us denote by $x|_I$ the image of x by the restriction r_I to coordinates in I :

$$r_I : \mathbb{R}_{\max}^n \rightarrow \mathbb{R}_{\max}^I, x \mapsto x|_I := (x_i)_{i \in I}. \quad (3.19)$$

We shall also use the notation

$$V|_I := \{v|_I \mid v \in V\}. \quad (3.20)$$

Lemma 3.3. *Let $I \subset [n]$ and denote*

$$M_I := \{y \in \mathbb{R}_{\max}^n \mid \text{Supp } y \subset I\}.$$

a) r_I is injective on M_I .
b) For all $y, z \in M_I$, we have

$$d(y, z) = \tilde{d}(y|_I, z|_I),$$

where in the right hand side \tilde{d} is the Hilbert projective distance on \mathbb{R}_{\max}^I .

c) If $y \in \mathbb{R}_{\max}^n, W \subset M_I$, then $d(y, W) = \tilde{d}(y|_I, W|_I)$.

Proof. a) Let $y', y'' \in M_I$ be such that $r_I(y') = r_I(y'')$, so $\text{Supp } y', \text{Supp } y'' \subset I$, $y'_i = y''_i$ ($i \in I$). Then $y'_j = y''_j = -\infty$ for all $j \notin I$, and hence $y' = y''$. Thus r_I is injective on M_I .

b) The second assertion follows from the fact that for $y, z \in M_I, \lambda \in \mathbb{R}_{\max}$, $y\lambda \leq z$ if and only if $y|_I\lambda \leq z|_I$. Indeed, we have

$$\begin{aligned} d(y, z) &= ((y \setminus z) \otimes (z \setminus y))^- \\ &= (\sup\{\lambda \in \mathbb{R}_{\max} | y\lambda \leq z\} \otimes \sup\{\mu \in \mathbb{R}_{\max} | z\mu \leq y\})^- \\ &= (\sup\{\lambda \in \mathbb{R}_{\max} | y|_I\lambda \leq z|_I\} \otimes \sup\{\mu \in \mathbb{R}_{\max} | z|_I\mu \leq y|_I\})^- \\ &= ((y|_I \setminus z|_I) \otimes (z|_I \setminus y|_I))^- = \tilde{d}(y|_I, z|_I). \end{aligned}$$

c) For the last assertion, let $w \in W$. Since then $w \in M_I$, from b) we get that $d(y, w) = \tilde{d}(y|_I, w|_I)$, whence, since $W|_I = r_I(W)$, we obtain

$$d(y, W) = \inf_{w \in W} d(y, w) = \inf_{w \in W} \tilde{d}(y|_I, w|_I) = \inf_{w' \in W|_I} \tilde{d}(y|_I, w') = \tilde{d}(y|_I, W|_I).$$

■

Proposition 3.2. Let V be a b-complete subsemimodule of \mathbb{R}_{\max}^n and $x \in \mathbb{R}_{\max}^n$ such that $d(x, V) < +\infty$. Define

$$x' := x|_{\text{Supp } x} \in \mathbb{R}^{\text{Supp } x}, V' := V^{(x)}|_{\text{Supp } x} \subset \mathbb{R}^{\text{Supp } x} \cup \{-\infty|_{\text{Supp } x}\}, \quad (3.21)$$

where $V^{(x)} := (V \cap [[x]]) \cup \{-\infty\}$ (of (3.15)). Then V' is a b-complete subsemimodule of $\mathbb{R}_{\max}^{\text{Supp } x} = \mathbb{R}^{\text{Supp } x} \cup \{-\infty|_{\text{Supp } x}\}$, $x' \in \mathbb{R}_{\max}^{\text{Supp } x}$, and we have

$$d(x, V) = \tilde{d}(x', V'), \quad (3.22)$$

where in the right hand side \tilde{d} is the Hilbert projective distance on $\mathbb{R}_{\max}^{\text{Supp } x}$. Furthermore, an element $v \in V$ is a best approximation of x in V if and only if $\text{Supp } v = \text{Supp } x$ and $v' := v|_{\text{Supp } x}$ is a best approximation of x' in V' .

Proof. Clearly $x' = x|_{\text{Supp } x} \in \mathbb{R}^{\text{Supp } x}$ and by Corollary 3.4a) we have $\text{Supp } v = \text{Supp } x$ for all $v \in [[x]]$, whence

$$V' = [(V \cap [[x]]) \cup \{-\infty\}]|_{\text{Supp } x} \subset \mathbb{R}^{\text{Supp } x} \cup \{-\infty|_{\text{Supp } x}\}.$$

Furthermore, since V is a b-complete subsemimodule of \mathbb{R}_{\max}^n , so is $V^{(x)}$ of (3.15) (by Proposition 3.1) and hence, since $V' = r_{\text{Supp } x}(V^{(x)})$, where $r_{\text{Supp } x} : \mathbb{R}_{\max}^n \rightarrow \mathbb{R}_{\max}^{\text{Supp } x}$ is a max-linear mapping, V' is a b-complete subsemimodule of $\mathbb{R}_{\max}^{\text{Supp } x}$.

By Proposition 3.1, we have $d(x, V) = d(x, V^{(x)})$ and the supports of the elements of $V^{(x)}$ are all included in $\text{Supp } x$, so by Lemma 3.3, we get that $d(x, V) = \tilde{d}(x', V')$, where \tilde{d} is the Hilbert projective distance on $\mathbb{R}_{\max}^{\text{Supp } x}$.

Assume now that $v \in V$ is a best approximation of x by V , that is, $d(x, v) = d(x, V)$. Then by Theorem 3.1, $v \in [[x]]$, whence by Corollary 3.4a), $\text{Supp } v = \text{Supp } x$; also, $v \in V \cap [[x]] \subset V^{(x)}$, whence $v|_{\text{Supp } x} \in V^{(x)}|_{\text{Supp } x} = V'$. Therefore, using Lemma 3.3, we obtain

$$\tilde{d}(x', v|_{\text{Supp } x}) = d(x, v) = d(x, V) = \tilde{d}(x', V'),$$

so $v' := v|_{\text{Supp } x}$ is a best approximation of x' in V' .

Conversely, assume now that $v \in \mathbb{R}_{\max}^n$ is such that $\text{Supp } v = \text{Supp } x$ and $v|_{\text{Supp } x}$ is a best approximation of x' in V' , that is, $\tilde{d}(x', v|_{\text{Supp } x}) = \tilde{d}(x', V')$. Then $v|_{\text{Supp } x} = r_{\text{Supp } x}(v)$ (by the injectivity of $r_{\text{Supp } x}$, see Lemma 3.3), whence $v \in V^{(x)} \subset V$, and using Lemma 3.3 we obtain

$$d(x, v) = \tilde{d}(x', v|_{\text{Supp } x}) = \tilde{d}(x', V') = d(x, V),$$

so v is a best approximation of x in V . ■

Remark 3.4. Denoting by n' the cardinality of $\text{Supp } x$ and using the isomorphism between $\mathbb{R}^{n'}$ and $\mathbb{R}^{\text{Supp } x}$, Proposition 3.2 shows that one can reduce the study of the best approximation of elements $x \in \mathbb{R}_{\max}^n$ by the elements of a b -complete subsemimodule V of \mathbb{R}_{\max}^n to the case (3.18). Practically, given V and $x \notin V$, whence also $d(x, V)$, if we want to find a best approximation of x by V , one can pass to $x' = x|_{\text{Supp } x}$ and $V' = V^{(x)}|_{\text{Supp } x}$, then find a best approximation v' of x' in V' , and then, by the above, the element $v = (v_1, \dots, v_n) \in V$ defined by

$$v_i = \begin{cases} v'_i & \text{if } i \in \text{Supp } x \\ -\infty & \text{if } i \notin \text{Supp } x \end{cases}$$

will be a best approximation of x by V .

4. FURTHER RESULTS ON THE UNIVERSAL SEPARATION THEOREM AND APPLICATIONS TO BEST APPROXIMATION

In classical linear analysis, one first reduces the problem of best approximation of elements x by linear subspaces V to the case of suitable half-spaces $H = H_{V,x}$ that separate V and x . In this section we shall apply a similar method to best approximation of $x \in \mathbb{R}_{\max}^n$ by elements of subsemimodules V of \mathbb{R}_{\max}^n . The relevant notion of half-space used for separation depends on the framework in which we are working. When considering best approximation by complete subsemimodules V of \mathbb{R}_{\max}^n , it is natural to use separation by *complete half-spaces* of \mathbb{R}_{\max}^n , while for best approximation by b -complete subsemimodules V of \mathbb{R}_{\max}^n it is natural to use separation by *closed half-spaces* of \mathbb{R}_{\max}^n , as we shall see below.

In [8], [9], [13] and [14], the separation theorems for \mathbb{R}_{\max}^n have been obtained as consequences of the results of [8] concerning complete semimodules. We shall follow here a similar approach, deducing the separation and best approximation results in \mathbb{R}_{\max}^n from separation and best approximation results in complete semimodules, since the proofs are more transparent in the latter setting.

Theorem 4.1. [8, Theorem 8] *Let X be a complete semimodule over the complete idempotent semiring \mathbb{S} . Let V be a complete subsemimodule of X , $x \in X$ and $x \notin V$, and consider the set*

$$K := \{h \in X \mid h \setminus x \leq h \setminus P_V(x)\} . \quad (4.1)$$

Then $V \subset K$ and $x \notin K$.

Remark 4.1. In [8], the result is written with the equality

$$h \setminus x = h \setminus P_V(x) \quad (4.2)$$

in (4.1); however, by a remark made in [14] for b -complete semimodules, which is valid also for complete semimodules, since $P_V(x) \leq x$, the inequality $h \setminus P_V(x) \leq h \setminus x$ holds for all $h \in X$, hence the two formulations are equivalent.

In [14], a *half-space* of a complete semimodule X is defined as a set of the form

$$K = K_{u,v} := \{h \in X \mid h \setminus u \leq h \setminus v\} , \quad (4.3)$$

with $u, v \in X$. Note that all half-spaces $K_{u,v}$ are complete subsemimodules of X . To be correct with the terminology “half-space”, one should avoid the case where $K = X$, which holds if and only if $u \leq v$, and the case where $K = \{\perp\}$ where \perp is the smallest element of X (which is also its neutral element for the addition \oplus). With this definition, the set K of Theorem 4.1 is a (complete) half-space, and when $x \notin V$, K separates x from V . We shall call it *the universal complete half-space of X separating x from V* .

Remark 4.2. In particular, if $\mathbb{S} = \overline{\mathbb{R}}_{\max}$, the complete max-plus semiring, and $X = \overline{\mathbb{R}}_{\max}^n$, (complete) half-spaces can be put in a more usual form, namely every complete half-space $K = K_{u,v}$ as in (4.3), with $u, v \in \overline{\mathbb{R}}_{\max}^n$, can be written in the form

$$H_{a,b} := \{h \in \overline{\mathbb{R}}_{\max}^n \mid ah \geq bh\} , \quad (4.4)$$

with $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \overline{\mathbb{R}}_{\max}^{1 \times n}$, where the notation ah means

$$ah = \max_{i \in [n]} (a_i + h_i) , \quad (4.5)$$

and conversely, every set $H = H_{a,b}$ as in (4.4) can be written in the form (4.3), with $u, v \in \overline{\mathbb{R}}_{\max}^n$. Indeed, by taking $a = -u^T, b = -v^T$, respectively

$u = -b^T, v = -a^T$, and using (2.13), we have

$$\begin{aligned} K_{u,v} &= \{h \in \overline{\mathbb{R}}_{\max}^n \mid \min_{i \in [n]}(u_i +' (-h_i)) \leq \min_{i \in [n]}(v_i +' (-h_i))\} \\ &= \{h \in \overline{\mathbb{R}}_{\max}^n \mid \max_{i \in [n]}(-u_i + h_i) \geq \max_{i \in [n]}(-v_i + h_i)\} \\ &= \{h \in \overline{\mathbb{R}}_{\max}^n \mid (-u^T)h \geq (-v^T)h\} = H_{-u^T, -v^T}. \end{aligned}$$

In [13], the universal separation theorem is written with half-spaces of the form $H_{a,b}$. Later we shall call a and b the “coefficients” of the representation (4.4) of H .

Since

$$\varphi_a(h) := ah \quad \forall h \in \overline{\mathbb{R}}_{\max}^n, \quad (4.6)$$

where $a \in \mathbb{R}_{\max}$, is the general form of the max-linear forms on $\overline{\mathbb{R}}_{\max}^n$ (e.g. by [8], Theorem 36; see also [17]), we can also write

$$H = \{h \in \overline{\mathbb{R}}_{\max}^n \mid \varphi_a(h) \geq \varphi_b(h)\}.$$

Remark 4.3. Let us mention that identifying $(\overline{\mathbb{R}}_{\max}^n)^*$ with $\overline{\mathbb{R}}_{\max}^n$ in the usual way, we may also regard ah of (4.5) as the “max-plus scalar product” of two row vectors or of two column vectors; however, we shall not use here this identification.

Our next aim will be to show that for an element x of a complete semimodule (respectively of a b -complete semimodule) X , the computation of the canonical projection onto, and the distance to, any (respectively any b -complete) subsemimodule V of X , can be reduced to the computation of the canonical projection onto, and the distance to, a complete half-space K (respectively a closed half-space H') of X .

For a subset M of any complete semimodule X over a complete idempotent semiring \mathbb{S} and any $x \in M$ let us set

$$\delta(x, M) := \sup_{v \in M} \delta(x, v); \quad (4.7)$$

then we may regard any $v_0 \in M$ satisfying

$$\delta(x, v_0) \geq \delta(x, v), \quad \forall v \in M, \quad (4.8)$$

(or, equivalently, $\delta(x, v_0) = \delta(x, M)$) as a “farthest point” in M from x , in the “anti-distance” δ .

Remark 4.4. Since the Hilbert projective distance d on X is defined by (2.20), the relation (4.8) is equivalent to $d(x, v_0)^- \geq d(x, v)^-$ ($v \in M$), that is, to $d(x, v_0) \leq d(x, v)$ ($v \in M$), meaning that v_0 is a nearest point in M to x in the Hilbert’s projective metric d . This remark will permit us to deduce results on nearest points in Hilbert’s projective metric d from results on farthest points in the anti-distance δ .

Theorem 4.2. [8, Theorem 18] *If V is a complete subsemimodule of a complete semimodule X over a complete idempotent semiring, and $x \in X$, then*

$$\delta(x, P_V(x)) \geq \delta(x, v), \quad \forall v \in V ,$$

i.e., $P_V(x)$ is a farthest point from x among the elements of V in the anti-distance δ . ■

We recall that the Hilbert's projective distance $d(x, V)$ between an element x and a set V is defined by (1.5).

Corollary 4.1. *If V is a complete subsemimodule of a complete semimodule X over a complete idempotent semiring, and if $x \in X$, then we have (1.9), or, in other words,*

$$d(x, P_V(x)) \leq d(x, v), \quad \forall v \in V ,$$

i.e., $P_V(x)$ is a best approximation of x in V for Hilbert's projective distance in X . ■

Remark 4.5. For $X = \overline{\mathbb{R}}_{\max}^n$, Corollary 4.1 has been given in [13], Theorem 1.

We next establish some additional properties of the universal separating complete half-space and apply them to reduce the problem of best approximation by subsemimodules to best approximation by half-spaces.

Theorem 4.3. *If V is a complete subsemimodule of a complete semimodule X over a complete idempotent semiring \mathbb{S} , if $x \notin V$, and if K is the associated complete half-space separating x and V (see Theorem 4.1), then*

$$P_V(x) = P_K(x).$$

Proof. Since $K \supset V$, we have $P_K(x) \geq P_V(x)$. If $h \in K$ is such that $h \leq x$, we have $\mathbf{1} \leq h \setminus x = h \setminus P_V(x)$, where $\mathbf{1}$ is the neutral element of \otimes in \mathbb{S} , and so, $h \leq P_V(x)$. Since this holds for all $h \in K$ such that $h \leq x$, it follows that $P_K(x) \leq P_V(x)$. ■

Corollary 4.2. *If V is a complete subsemimodule of a complete semimodule X over a complete idempotent semiring, if $x \in X, x \notin V$, and if K is the associated complete half-space (4.1) separating x and V , then we have*

$$d(x, V) = d(x, K) .$$

Proof. Combining Corollary 4.1 and Theorem 4.3, we obtain

$$d(x, V) = d(x, P_V(x)) = d(x, P_K(x)) = d(x, K). \blacksquare$$

Finally, let us show the connection between the canonical projection and orthogonality. The relation (4.2) can be thought of as an analogue of the classical orthogonality relation $\langle h, x - P_V(x) \rangle = 0$, where $\langle ., . \rangle$ denotes the usual inner product, characterizing the nearest point $P_V(x)$ of an element x onto a linear subspace. We next show that in the setting of semimodules, the canonical projection $P_V(x)$ is still characterized by the previous “orthogonality” property.

Definition 4.1. If X is a complete idempotent semimodule, for $x, y, z \in X$ we shall say that the “bivector” $(x, y) \in X^2$ is *orthogonal to* z , and we shall write $(x, y) \perp z$, if

$$z \setminus x = z \setminus y. \quad (4.9)$$

The bivector $(x, y) \in X^2$ is said to be *orthogonal to a subset* M of X , and we write $(x, y) \perp M$, if $(x, y) \perp z$ for all $z \in M$.

In particular, if $X = \overline{\mathbb{R}}_{\max}^n$ and $x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T, z = (z_1, \dots, z_n)^T \in \overline{\mathbb{R}}_{\max}^n$, then by (2.13), the relation (4.9) is equivalent to

$$\wedge_{i \in [n]} (x_i +' (-z_i)) = \wedge_{i \in [n]} (y_i +' (-z_i)).$$

Theorem 4.1 shows that for any complete subsemimodule V of $\overline{\mathbb{R}}_{\max}^n$ and any $x \notin V$, the bivector $(x, P_V(x))$ is orthogonal to V . Now we shall show that $P_V(x)$ is the only element of V with this property.

Theorem 4.4. Let V be a complete subsemimodule of a complete semimodule X over a complete idempotent semiring, and let $x \in X, x \notin V$. Then, $P_V(x)$ is the unique element y of V such that $(x, y) \perp V$, i.e., such that

$$v \setminus x = v \setminus y, \quad \forall v \in V. \quad (4.10)$$

Proof. By Theorem 4.1, $y = P_V(x)$ satisfies the above relations. We next show that y is unique.

If y satisfies (4.10), then for all $v \in V$ we have $y \geq v(v \setminus y) = v(v \setminus x)$, and so, $y \geq P_V(x) = \sup_{v \in V} v(v \setminus x)$.

Moreover, taking $v = y$ in (4.10), we get $y \setminus x = y \setminus y \geq \mathbf{1}$, where $\mathbf{1}$ is the neutral element of \mathbb{S} for \otimes , and so $x \geq y$. Since $P_V(x)$ is the maximal element of V which is bounded above by x , it follows that $y \leq P_V(x)$. Hence $y = P_V(x)$. ■

Let us pass now to $\overline{\mathbb{R}}_{\max}^n$. As mentioned at the beginning of this section, when considering the b -complete (but not complete) semimodule $\overline{\mathbb{R}}_{\max}^n$, instead of $\overline{\mathbb{R}}_{\max}^n$, one is rather interested to take the *closed half-spaces* of $\overline{\mathbb{R}}_{\max}^n$ as tools for separation, which are defined as the sets of the form

$$\begin{aligned} H' = H'_{a,b} &= \{h \in \overline{\mathbb{R}}_{\max}^n \mid ah \geq bh\} \\ &= \{h \in \overline{\mathbb{R}}_{\max}^n \mid \max_{i \in [n]} (a_i + h_i) \geq \max_{i \in [n]} (b_i + h_i)\}, \end{aligned} \quad (4.11)$$

where $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in (\overline{\mathbb{R}}_{\max}^n)^*$ are row vectors with coordinates in $\overline{\mathbb{R}}_{\max}$. We will call H' the *universal closed half-space of $\overline{\mathbb{R}}_{\max}^n$ separating x from V* . The term “closed” refers to the usual topology of $\overline{\mathbb{R}}_{\max}^n$, since the set H' of (4.11) with $a, b \in (\overline{\mathbb{R}}_{\max}^n)^*$ is always closed in $\overline{\mathbb{R}}_{\max}^n$ (by [9], Proposition 3.7). A particular case which will be important in the sequel is that when $H'_{a,b}$ has finite apex (we recall that the number $-(a \oplus b)$ is called [15] the *apex* of $H'_{a,b}$).

Any closed half-space $H'_{a,b}$ of $\overline{\mathbb{R}}_{\max}^n$ is the trace over $\overline{\mathbb{R}}_{\max}^n$ of a complete half-space of $\overline{\mathbb{R}}_{\max}^n$ (but not vice versa). Indeed, taking $X = \overline{\mathbb{R}}_{\max}^n$ thought of

as a complete $\overline{\mathbb{R}}_{\max}$ -semimodule, and taking $u = -a^T$ and $v = -b^T$, where $a, b \in (\mathbb{R}_{\max}^n)^*$, by Remark 4.2 we obtain

$$K_{u,v} \cap \mathbb{R}_{\max}^n = H_{a,b} \cap \mathbb{R}_{\max}^n = H'_{a,b} .$$

For b -complete subsemimodules of \mathbb{R}_{\max}^n we obtain the following results:

Corollary 4.3. *If V is a b -complete subsemimodule of \mathbb{R}_{\max}^n , and if $x \in \mathbb{R}_{\max}^n$, then*

$$d(x, P_V(x)) \leq d(x, v), \quad \forall v \in V ,$$

i.e., $P_V(x)$ is a best approximation of x in V for Hilbert's projective distance in \mathbb{R}_{\max}^n .

Proof. Since V is a b -complete subsemimodule of \mathbb{R}_{\max}^n , it has a completion \hat{V} in $\overline{\mathbb{R}}_{\max}^n$, which consists of the suprema of arbitrary subsets of V . The latter is a complete subsemimodule of $\overline{\mathbb{R}}_{\max}^n$. It is readily seen that $P_{\hat{V}}(x) = P_V(x)$, so the result follows from Corollary 4.1. ■

Corollary 4.4. *If V is a b -complete subsemimodule of \mathbb{R}_{\max}^n , if $x \in \mathbb{R}_{\max}^n, x \notin V$, and if*

$$\begin{aligned} H' &= \{h \in \mathbb{R}_{\max}^n \mid h \setminus x \leq h \setminus P_V(x)\} \\ &= \{h \in \mathbb{R}_{\max}^n \mid \max_{j \in [n]} (h_j - x_j) \geq \max_{j \in [n]} (h_j - P_V(x)_j)\}, \end{aligned} \tag{4.12}$$

then

$$P_V(x) = P_{H'}(x).$$

Here, and in the sequel, for $a, b \in \overline{\mathbb{R}}_{\max}$, we set

$$a - b := a + (-b) .$$

Proof. This follows similarly to Corollary 4.3, using now Theorem 4.3. ■

Corollary 4.5. *If V is a b -complete subsemimodule of \mathbb{R}_{\max}^n , and if $x \in \mathbb{R}_{\max}^n, x \notin V$, then*

$$d(x, V) = d(x, H') ,$$

with H' of (4.12).

Proof. Applying Corollaries 4.3 and 4.4, we get

$$d(x, V) = d(x, P_V(x)) = d(x, P_{H'}(x)) = d(x, H') . \blacksquare$$

5. THE CANONICAL PROJECTION ONTO, AND THE DISTANCE TO, A CLOSED HALF-SPACE OF \mathbb{R}_{\max}^n

In the next result we shall give an explicit formula for the canonical projection onto a closed half-space of \mathbb{R}_{\max}^n . To this end, the following notation will be useful: If $b \in (\mathbb{R}_{\max}^n)^*$ is a row vector and $\lambda \in \mathbb{R}_{\max}$ is a scalar, we set

$$b \setminus \lambda := \sup\{u \in \mathbb{R}_{\max}^n \mid bu \leq \lambda\} \in \overline{\mathbb{R}}_{\max}^n, \tag{5.1}$$

u being thought of as a column vector. So $b \setminus \lambda$ is a column vector with entries

$$\begin{aligned} (b \setminus \lambda)_j &= (\sup\{u \in \mathbb{R}_{\max}^n \mid bu \leq \lambda\})_j \\ &= \sup\{u \in \mathbb{R}_{\max}^n \mid b_j u \leq \lambda\} = b_j \setminus \lambda = \begin{cases} (b_j)^{-1}\lambda & \text{if } j \in \text{Supp } b \\ +\infty & \text{if } j \notin \text{Supp } b. \end{cases} \end{aligned} \quad (5.2)$$

Theorem 5.1. *Let $a, b \in (\mathbb{R}_{\max}^n)^*$ be row vectors and consider the closed half-space*

$$H = \{h \in \mathbb{R}_{\max}^n \mid ah \geq bh\}. \quad (5.3)$$

Let $I = \text{Supp } a$, $J = \text{Supp } b$, and assume $I \cap J = \emptyset$ and that $J \neq \emptyset$ ($b \neq -\infty$). Then for any $x \in \mathbb{R}_{\max}^n$ we have

$$P_H(x) = x \wedge (b \setminus ax), \quad (5.4)$$

i.e.,

$$\begin{aligned} (P_H(x))_j &= x_j \wedge (b_j \setminus ax) \\ &= \begin{cases} x_j & \text{for } j \in J^c, \\ x_j \wedge (b_j^{-1}(ax)) = x_j \wedge (b_j^{-1}(\bigoplus_{i \in I} a_i x_i)) & \text{for } j \in J, \end{cases} \end{aligned} \quad (5.5)$$

where J^c denotes the complement of J in $[n]$.

Proof. We set

$$u := x \wedge (b \setminus ax)$$

and first observe that the coordinates of u coincide with the right hand side of (5.5).

Assume that $h \in H$ is such that $x \geq h$. Then, $ax \geq ah \geq bh$, and so, $h \leq \sup\{u' \in \mathbb{R}_{\max}^n \mid bu' \leq ax\} = b \setminus ax$. It follows that $h \leq x \wedge (b \setminus ax) = u$. This implies that $P_H(x) = \sup\{h \in H \mid h \leq x\} \leq u$. To show that the equality holds, it remains to check that $au \geq bu$. We have

$$bu = b(x \wedge (b \setminus ax)) \leq b(b \setminus ax) = b \sup\{u \in \mathbb{R}_{\max}^n \mid bu \leq ax\} \leq ax = \bigoplus_{i \in I} a_i x_i.$$

But by $I \cap J = \emptyset$ we have $I \subseteq J^c$, whence by (5.5), $a_i x_i = a_i x_i (b_i \setminus ax) = a_i u_i$ ($i \in I$), and therefore $\bigoplus_{i \in I} a_i x_i = \bigoplus_{i \in I} a_i u_i = au$. Thus, finally, $bu \leq au$. ■

The following result gives the main formula for the distance to a closed half-space:

Theorem 5.2. *Let $a, b \in (\mathbb{R}_{\max}^n)^*$ be row vectors, H the closed half-space (5.3), and $x \notin H$. Then*

$$d(x, H) = ax \setminus bx = \begin{cases} (ax)^{-1}bx & \text{if } ax \neq -\infty, \\ +\infty & \text{if } ax = -\infty. \end{cases} \quad (5.6)$$

Proof. Since $P_H(x)$ maximizes the opposite of Hilbert's distance to x among the points of H , see Theorem 4.2, we have

$$\delta(x, H) = \delta(x, P_H(x)) = (P_H(x) \setminus x)(x \setminus P_H(x)) . \quad (5.7)$$

Assume first that $P_H(x) \neq -\infty$. We claim that in that case we have $P_H(x) \setminus x = 0$. Indeed, since $P_H(x) \leq x$, we must have $\lambda := P_H(x) \setminus x \geq 0$. Assume by contradiction that $\lambda > 0$. Then since $P_H(x) \neq -\infty$, $P_H(x)\lambda > P_H(x)0 = P_H(x)$. Since H is a max-plus linear subspace, and since $P_H(x) \in H$, we have $P_H(x)\lambda \in H$, but since by the definition of λ , $P_H(x)\lambda \leq x$, this contradicts the definition of $P_H(x)$ as the maximal element $h \in H$ such that $h \leq x$.

Then, using successively Equations (5.7), (5.4), and residuation properties of max-plus linear maps (see [8]), we get

$$\begin{aligned} \delta(x, H) &= (P_H(x) \setminus x)(x \setminus P_H(x)) = x \setminus P_H(x) \\ &= x \setminus (x \wedge (b \setminus ax)) = (x \setminus x) \wedge (x \setminus (b \setminus ax)) \\ &= (x \setminus x) \wedge (bx \setminus ax) . \end{aligned} \quad (5.8)$$

Since $x \notin H$, we have $x \neq -\infty$ (because $-\infty \in \{h \in \mathbb{R}_{\max}^n \mid ah \geq bh\} = H$), so $x \setminus x = 0$. Also, again since $x \notin H$, we have $bx > ax$. Hence $bx \neq -\infty$ and $bx \setminus ax < 0$, so (5.8) simplifies to

$$\delta(x, H) = 0 \wedge (bx \setminus ax) = bx \setminus ax = (bx)^{-1}ax .$$

Consequently, by (2.20), we arrive at

$$d(x, H) = (\delta(x, H))^{-1} = ax \setminus bx .$$

Assume now that $P_H(x) = -\infty$. Then since $x \notin H$, so $x \neq -\infty$, we have, using (1.9), that $d(x, H) = d(x, P_H(x)) = d(x, -\infty) = +\infty$. Moreover, by (5.5), we get that $x_i = P_H(x)_i = -\infty$ for all $i \notin J$, so that in particular $a_i x_i = -\infty$ ($i \notin J$). Hence by the definition of J and since $bx > ax$, we get that $ax \setminus bx = \sup\{\lambda \in \mathbb{R}_{\max} \mid \lambda ax \leq bx\} = +\infty = d(x, H)$. ■

6. THE CANONICAL FORMS OF CLOSED HALF-SPACES OF \mathbb{R}_{\max}^n

We have the following result, which shows that every closed half-space (4.11) of \mathbb{R}_{\max}^n admits a canonical representation with the aid of coefficients with disjoint supports:

Proposition 6.1. *Let $a, b \in (\mathbb{R}_{\max}^n)^* \setminus \{-\infty\}$ be row vectors such that $a \not\geq b$ and there exists $i \in [n]$ such that $a_i \geq b_i$, and consider the closed half space*

$$H = \{h \in \mathbb{R}_{\max}^n \mid ah \geq bh\} \quad (6.1)$$

(the assumptions on the coefficients a and b are equivalent to $\{-\infty\} \neq H \neq \mathbb{R}_{\max}^n$). Let a' and $b' \in (\mathbb{R}_{\max}^n)^*$ be the truncations of a and b defined by

$$a'_i = \begin{cases} a_i & \text{if } a_i \geq b_i \\ -\infty & \text{if } a_i < b_i, \end{cases} \quad b'_j = \begin{cases} b_j & \text{if } a_j < b_j \\ -\infty & \text{if } a_j \geq b_j. \end{cases} \quad (6.2)$$

Then $\text{Supp } a' \cap \text{Supp } b' = \emptyset$, and H can be written in the form:

$$H = \{h \in \mathbb{R}_{\max}^n \mid a'h \geq b'h\} . \quad (6.3)$$

Proof. Let us denote

$$J := \{j \in [n] \mid a_j < b_j\}, \quad J^c := \{j \in [n] \mid a_j \geq b_j\},$$

so that

$$a'_i = \begin{cases} a_i & \text{for } i \in J^c \\ -\infty & \text{otherwise,} \end{cases} \quad b'_j = \begin{cases} b_j & \text{for } j \in J \\ -\infty & \text{otherwise.} \end{cases} \quad (6.4)$$

Thus, $\text{Supp } (a') \subseteq J^c$ and $\text{Supp } (b') \subseteq J$, whence $\text{Supp } (a') \cap \text{Supp } (b') = \emptyset$.

Furthermore, let H' be the right hand side of (6.3), and let us show that $H = H'$. The elements a' and b' satisfy $a' \leq a$ and $b' \leq b$ and since $a'_i = a_i \geq b_i$ for $i \in J^c$, and $b'_i = b_i$ for $i \in J$, we deduce that $b \leq a' \oplus b'$.

Let $h \in H'$, then $b'h \leq a'h$. Hence $bh \leq (a' \oplus b')h \leq a'h \leq ah$, so $h \in H$, which shows the inclusion $H' \subseteq H$.

Conversely, let $h \in H$, then $bh \leq ah$. Since $b' \leq b$, this implies that $b'h \leq ah$. Let a'' be the truncation of a to J , then $a = a' \oplus a''$ and thus

$$b'h \leq a'h \oplus a''h . \quad (6.5)$$

If the support of h does not intersect J , then $a''h = -\infty$ and (6.5) implies that $b'h \leq a'h$, that is $h \in H'$. Otherwise, since $a_i < b_i$ for all $i \in J$, we deduce that $a''h < b'h$, hence by (6.5), it follows that the maximum of $a'h$ and $a''h$ which is greater or equal to $a'h$, is necessarily equal to $a'h$. Hence again $b'h \leq a'h$, and thus $h \in H'$. We have shown the converse inclusion $H \subseteq H'$, hence the equality. ■

Corollary 6.1. *Let a, b and H be as in Proposition 6.1 and assume that $x \notin H$. Then,*

$$d(x, H) = a'x \setminus bx ,$$

where a' is defined as in Proposition 6.1.

Proof. Using Theorem 5.2 for the coefficients a', b' defined in Proposition 6.1, we get that $d(x, H) = a'x \setminus b'x$. Since $x \notin H$, we have $ax < bx$. Let J and b' be defined as in Proposition 6.1. Since $b_i \leq a_i$ when $i \in J^c$, we deduce that $bx = b'x$, which shows the corollary. ■

Definition 6.1. We shall call (6.3) the *canonical form* of the closed half-space H .

For the computation of distances to, and elements of best approximation by, subsemimodules, it is worthwhile to write explicitly the canonical form of the universal separating closed half-space (4.12) for a pair (V, x) , where V is a b -complete subsemimodule of \mathbb{R}_{\max}^n and $x \notin V$:

Corollary 6.2. *If V is a b-complete subsemimodule of \mathbb{R}_{\max}^n and $x \in \mathbb{R}_{\max}^n$, $x \notin V$ is such that all coordinates of $P_V(x)$ (and hence also of x) are $> -\infty$, then the following closed half-space separates x from V :*

$$\begin{aligned} H'_{V,x} &= \{h \in \mathbb{R}_{\max}^n \mid \max_{j|x_j=P_V(x)_j} (h_j - x_j) \geq \max_{j|x_j>P_V(x)_j} (h_j - (P_V(x)_j))\} \\ &= \{h \in \mathbb{R}_{\max}^n \mid \wedge_{j \in J} h_j \setminus x_j \leq \wedge_{j \in J^c} h_j \setminus P_V(x)_j\}, \end{aligned} \quad (6.6)$$

where

$$J = \{j \in [n] \mid x_j = P_V(x)_j\}, \quad J^c = \{j \in [n] \mid x_j > P_V(x)_j\}. \quad (6.7)$$

Proof. This follows from Proposition 6.1, setting

$$a_j = -x_j, \quad b_j = -P_V(x)_j \quad (j \in [n]).$$

Indeed, then $a_j \geq b_j \Leftrightarrow -x_j \geq P_V(x)_j \Leftrightarrow x_j = P_V(x)_j$ (where the last equivalence holds by $P_V(x) \leq x$) and $a_j < b_j \Leftrightarrow -x_j < -P_V(x)_j \Leftrightarrow x_j > P_V(x)_j$, whence by (6.2),

$$a'_j = \begin{cases} -x_j & \text{if } x_j = P_V(x)_j \\ -\infty & \text{if } x_j > P_V(x)_j, \end{cases} \quad b'_j = \begin{cases} -P_V(x)_j & \text{if } x_j > P_V(x)_j \\ -\infty & \text{if } x_j = P_V(x)_j. \end{cases} \quad (6.8)$$

Consequently, $a'h = \max_{j|x_j=P_V(x)_j} (h_j - x_j)$ and $b'h = \max_{j|x_j>P_V(x)_j} (h_j - P_V(x)_j)$, whence by (6.3) we obtain

$$\begin{aligned} H &= \{h \in \mathbb{R}_{\max}^n \mid a'h \geq b'h\} \\ &= \{h \in \mathbb{R}_{\max}^n \mid \max_{j|x_j=P_V(x)_j} (h_j - x_j) \geq \max_{j|x_j>P_V(x)_j} (h_j - (P_V(x)_j))\} \\ &= H'_{V,x}. \blacksquare \end{aligned}$$

Remark 6.1. a) In the above, since $x \notin V$, we have $J^c \neq \emptyset$. Furthermore, we also have $J \neq \emptyset$, since otherwise $P_V(x)_j < x_j$ ($j \in [n]$), whence by (6.6) we would obtain $H'_{V,x} = \emptyset$.

Note also that the coefficients $-x_j$ and $-P_V(x)_j$ in the canonical form (6.6) of $H'_{V,x}$ depend on V and x , while the coefficients a'_j, b'_j in the canonical form (6.3) of (6.1) don't.

b) The assumption alone that all coordinates of x are $> -\infty$ does not imply that each element v of V has all coordinates $> -\infty$, as shown e.g. by the subsemimodule $V = \{(-\infty, v_2) \mid v_2 \in \mathbb{R}\}$ of \mathbb{R}_{\max}^2 .

c) Corollary 6.2 is a more precise form of [13], Theorem 3.

By (6.7), (6.8) and the assumption that all $P_V(x)_j$ are $> -\infty$, we have $\text{Supp}(a') = J$ and $\text{Supp}(b') = J^c$, and hence in the situation of Corollary 6.2 we always have

$$\text{Supp}(a') \cup \text{Supp}(b') = J \cup J^c = [n]. \quad (6.9)$$

Definition 6.2. We shall call the sets H' of the form (6.3) satisfying $\text{Supp } a' \cap \text{Supp } b' = \emptyset$ and (6.9), *half-spaces with finite apex*.

Note that the sets of this form are exactly the “tropical half-spaces” studied in [15], where the apex of the half-space (6.3) is defined as the vector $-(a' \oplus b')$.

Remark 6.2. In classical linear analysis, one first reduces the problem of best approximation of elements x by linear subspaces V to the case of suitable separating support half-spaces $H = H_{V,x}$ by showing for them the equality of distances $d(x, V) = d(x, H)$ and the equality of elements of best approximation in V and H , then one solves the problems of best approximation for general half-spaces H , and this gives solutions also for the problems of best approximation by the linear subspaces V . In the case of best approximation of x by elements of subsemimodules V of \mathbb{R}_{\max}^n such that all coordinates of $P_V(x)$ (and hence also of x) are $> -\infty$, in order to apply such a method one needs to use closed half-spaces with finite apex, as shown by Corollary 6.2.

The following immediate consequence of Corollary 6.2 shows that the *sectors* of H' , as defined in [15] are readily obtained from the previous representation, and that the apex of H' is precisely $P_V(x)$.

Corollary 6.3. *Let x, V and H' be as in Corollary 6.2. Then, the apex of the half-space H' is $P_V(x)$, and H' is the union of the sectors*

$$H'_i := \{x \in \mathbb{R}_{\max}^n \mid h_i - (P_V(x))_i \geq \max_{j \in [n] \setminus \{i\}} (h_j - (P_V(x))_j)\} \quad \forall i \in I .$$

■

In the above the term “closed half-space” was introduced because of the analogy with the classical closed half-spaces $\{x \in \mathbb{R}^n \mid \Phi(x) \leq c\}$ of \mathbb{R}^n , where $\Phi \in (\mathbb{R}^n)^*$, $c \in \mathbb{R}$. However, note that there is an important difference between the two cases. Namely, in the classical case of \mathbb{R}^n , given a linear subspace V of \mathbb{R}^n and a point $x \notin V$, there exists a separating closed half-space $H = H_{V,x}$ of \mathbb{R}^n (i.e. such that $V \subseteq H, x \notin H$), with the additional property $d(x, V) = d(x, H)$, but for any other separating closed half-space $H' \neq H$ ($V \subset H', x \notin H'$) we must have $H' \subset H$ (strictly) and hence $d(x, H') < d(x, H)$, because $\text{bd } H'$ must be parallel to $\text{bd } H$ (these facts are well known and easy to prove). However, this fact is no longer true in the case of closed half-spaces $H = V, H'$ and outside points $x \notin H'$ in \mathbb{R}_{\max}^n , as shown by Example 6.1 below, in which $H' \supset H$, $H' \neq H$, $d(x, H') = d(x, H)$:

Example 6.1. Let

$$\begin{aligned} H &= V := \{v \in \mathbb{R}_{\max}^3 \mid v_2 \geq v_1\} \\ &= \{v \in \mathbb{R}_{\max}^3 \mid (-\infty)v_1 \oplus 0v_2 \oplus (-\infty)v_3 \geq 0v_1 \oplus (-\infty)v_2 \oplus (-\infty)v_3\}, \\ x &:= (2, 1, 0)^T \notin V. \end{aligned}$$

Then V is a subsemimodule (actually a half-space, but not with finite apex), and

$$\begin{aligned} P_V(x) &= \max\{v \in V | (v_1, v_2, v_3)^T \leq (2, 1, 0)^T\} = (1, 1, 0)^T, \\ J &= \{j | x_j = P_V(x)_j\} = \{2, 3\}, J^c = \{j | x_j > P_V(x)_j\} = \{1\}, \end{aligned}$$

so $J \cup J^c = [3]$, and hence the universal separating closed half-space H' of (6.6) has finite apex; in fact,

$$\begin{aligned} H' &= H'_{V,x} = \{h \in \mathbb{R}_{\max}^3 | \max(-x_2 + h_2, -x_3 + h_3) \geq -P_V(x)_1 + h_1\} \\ &= \{h | \max(-1 + h_2, 0 + h_3) \geq -1 + h_1\} = \{h | \max(h_2, h_3 + 1) \geq h_1\}. \end{aligned}$$

Furthermore, we have $d(x, V) = d(x, H')$ and $H \subset H'$ (strictly). This is illustrated in Figure 1, in which every max-plus line through the origin (i.e. the set of multiples of a vector of \mathbb{R}_{\max}^3) is represented by its intersection point with a hyperplane orthogonal to the main diagonal.

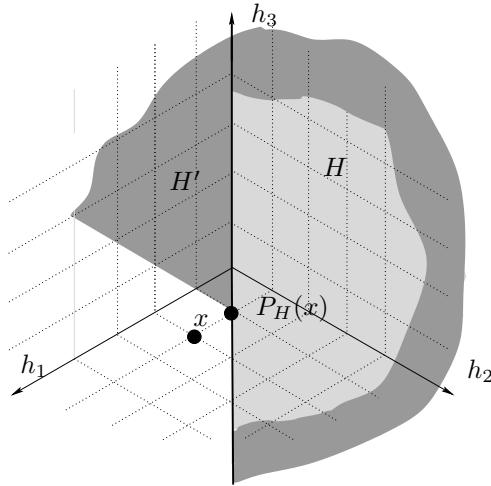


FIGURE 1. The half-space $H = \{h \in \mathbb{R}_{\max}^3 \mid h_2 \geq h_1\}$ (light gray). The universal separating closed half-space H' with apex $P_H(x)$ (dark gray), see Example 6.1.

7. THE ELEMENTS OF BEST APPROXIMATION BY CLOSED HALF-SPACES

By the above results, the problem of best approximation by subsemimodules of \mathbb{R}_{\max}^n can be reduced to that of best approximation by closed half-spaces with finite apex. In the present section, more generally, we give characterizations of the elements of best approximation by arbitrary closed half-spaces in \mathbb{R}_{\max}^n (that are not assumed to have finite apex). If $a \in (\mathbb{R}_{\max}^n)^*$ is a row vector and $x \in \mathbb{R}_{\max}^n$ a column vector, we define

$$\text{Argmax}(a, x) := \{i \in [n] \mid a_i x_i = ax\} , \quad (7.1)$$

which is always a nonempty set. The following is clear:

$$ax \neq -\infty \Rightarrow \text{Argmax}(a, x) \subset \text{Supp } a \cap \text{Supp } x . \quad (7.2)$$

The next theorem gives an analytic characterization of the set of elements of best approximation.

Theorem 7.1. *Let $a, b \in (\mathbb{R}_{\max}^n)^*$ be row vectors, H the closed half-space (5.3), and assume that the sets*

$$I := \text{Supp } a, \quad J := \text{Supp } b, \quad (7.3)$$

satisfy $I \cap J = \emptyset$ and $J \neq \emptyset$ ($b \neq -\infty$). Furthermore, let $x \in \mathbb{R}_{\max}^n$, $x \notin H$ be such that $d(x, H) < +\infty$. For an element $h \in \mathbb{R}_{\max}^n$ the following assertions are equivalent :

1° . *h is a best approximation of x in the closed half-space H ;*

2° . *$ah \geq bh \neq -\infty$ and*

$$x(bx)^{-1}(ah) \leq h \leq x(ax)^{-1}(bh); \quad (7.4)$$

3° . *There exist $\lambda \neq -\infty$ and $i \in \text{Argmax}(a, x)$ such that the following conditions hold:*

$$a_i h_i = \lambda , \quad (7.5)$$

$$b_j h_j = \lambda , \quad \forall j \in \text{Argmax}(b, x) , \quad (7.6)$$

$$x_k(bx)^{-1}\lambda \leq h_k \leq (P_H(x))_k(ax)^{-1}\lambda , \quad (7.7)$$

$$\forall k \in [n] \setminus (\text{Argmax}(b, x) \cup \{i\}) ;$$

moreover, in this case λ is unique, namely $\lambda = ah = bh$.

Proof. By Theorem 5.2 and our assumption, $d(x, H) = ax \setminus bx < +\infty$, and so $ax \neq -\infty$. Furthermore, since $x \notin H$, we have $bx > ax$, and in particular $bx \neq -\infty$.

$1^\circ \implies 2^\circ$. Let h be a best approximation of x in H , that is, $h \in H$ (so $ah \geq bh$) and $d(x, h) = d(x, H)$, which is equivalent to the condition $\delta(x, h) \geq \delta(x, H)$, that is,

$$(x \setminus h)(h \setminus x) \geq ax(bx)^{-1} . \quad (7.8)$$

Since $d(x, h) = d(x, H) < +\infty$, x and h must have the same support (by Lemma 3.2). Then, since $ax \neq -\infty$ we deduce that $ah \neq -\infty$ (indeed, there is at least one index i such that $a_i x_i = ax \neq -\infty$, and so $x_i \neq -\infty$; hence, since x and h have the same support, $h_i \neq -\infty$, and so $ah \geq a_i h_i \neq -\infty$). Similarly, we deduce from $bx \neq -\infty$, that $bh \neq -\infty$. Furthermore, (7.8) implies that $x \setminus h \geq (h \setminus x)^{-1}(ax)(bx)^{-1}$ or equivalently (see (2.15)),

$$h \geq x(h \setminus x)^{-1}(ax)(bx)^{-1} . \quad (7.9)$$

Similarly, from (7.8) one also obtains

$$x(x \setminus h)(ax)^{-1}(bx) \geq h . \quad (7.10)$$

Since $h\lambda \leq x$ implies $ah\lambda \leq ax$, it follows that $h \setminus x \leq \sup\{\lambda | ah\lambda \leq ax\} = (ah) \setminus (ax) = (ah)^{-1}(ax)$, whence $(h \setminus x)^{-1} \geq (ax)^{-1}ah$. Using this inequality

in (7.9), we get $h \geq x(ah)(bx)^{-1}$, which is the first inequality in (7.4). Furthermore, $x \setminus h \leq (bx) \setminus (bh) = (bx)^{-1}(bh)$ and together with (7.10), this implies that $h \leq x(bh)(ax)^{-1}$, which is the second inequality in (7.4). This completes the proof of the implication $1^\circ \implies 2^\circ$.

$2^\circ \implies 1^\circ$. Let h be as in 2° . Then by $ah \geq bh$ we have $h \in H$. Using (2.15), from the first inequality in (7.4) we obtain that $x \setminus h \geq (bx)^{-1}(ah)$. By the second inequality in (7.4), and the fact that $bh \neq -\infty$, we obtain that $h(bh)^{-1}(ax) \leq x$, which implies, using (2.15), that $h \setminus x \geq (bh)^{-1}(ax)$. Hence $(h \setminus x)(x \setminus h) \geq (bx)^{-1}(ah)(bh)^{-1}(ax)$ and since $ah \geq bh$, we obtain $(h \setminus x)(x \setminus h) \geq (bx)^{-1}(ax)$, that is, (7.8), which itself is equivalent to the condition $d(x, h) = d(x, H)$.

$2^\circ \implies 3^\circ$. Let h be as in 2° , set $\lambda = bh$, and pick some $i \in \text{Argmax}(a, h)$ (the latter set is necessarily nonempty). We shall see later that

$$\text{Argmax}(a, h) \subset \text{Argmax}(a, x) , \quad (7.11)$$

so that $i \in \text{Argmax}(a, x)$ as requested in 3° .

By 2° , we must have $\lambda = bh \in \mathbb{R}$. Multiplying the first inequality of (7.4) by b , or the second one by a , we deduce that $ah \leq bh$, and since $ah \geq bh$ also holds by 2° , we get that $ah = bh = \lambda$. Consequently, since by our choice $i \in \text{Argmax}(a, h)$, we have $a_i h_i = \lambda$.

Using the fact that $ah = bh = \lambda$, we deduce from (7.4) that

$$x_k(bx)^{-1}\lambda \leq h_k \leq x_k(ax)^{-1}\lambda, \quad \forall k \in [n] . \quad (7.12)$$

Furthermore, since $b_k h_k \leq bh = \lambda$, we deduce $h_k \leq b_k \setminus \lambda = (b_k \setminus (ax))(ax)^{-1}\lambda$, for all $k \in [n]$. This, together with the second inequality in (7.12) and formula (5.4) for $P_H(x)$, implies

$$h_k \leq (x_k \wedge b_k \setminus (ax))(ax)^{-1}\lambda = (P_H(x))_k(ax)^{-1}\lambda .$$

Together with the first inequality in (7.7), this establishes the inequalities (7.7) for all $k \in [n]$, and a fortiori for all $k \in [n] \setminus (\text{Argmax}(b, x) \cup \{i\})$.

Now we show (7.11). By the second part of (7.4) we have $a_k h_k \leq a_k x_k (ax)^{-1}\lambda$ for all $k \in [n]$, and hence for any k such that $a_k x_k < ax$, we have $a_k h_k \leq a_k x_k (ax)^{-1}\lambda < \lambda = ah$, whence $k \notin \text{Argmax}(a, h)$, which shows (7.11). Since we already proved that $a_i h_i = \lambda$, we deduce (7.5).

Finally, if $j \in \text{Argmax}(b, x)$, that is, $b_j x_j = bx$, then $\lambda = bx(bx)^{-1}\lambda = b_j x_j (bx)^{-1}\lambda \leq b_j h_j \leq bh = \lambda$ (where the penultimate inequality is obtained by multiplying by b_j the first inequality of (7.7) for $k = j$), whence we obtain (7.6).

$3^\circ \implies 2^\circ$. Let h, i and λ be as in 3° .

We claim that the inequalities in (7.7) are valid for all $k \in [n]$.

Indeed, since $i \in \text{Argmax}(a, x)$, then by $a_i x_i = ax < bx$ we have

$$a_i x_i (bx)^{-1}\lambda < bx(bx)^{-1}\lambda = \lambda = a_i h_i = a_i x_i (ax)^{-1}\lambda .$$

Moreover, since $a_i h_i = \lambda \in \mathbb{R}$, we have $a_i \neq -\infty$, and so $x_i(bx)^{-1}\lambda \leq h_i \leq x_i(ax)^{-1}\lambda$. Since $i \in I \subset J^c$ by assumption, hence $b_i \setminus (ax) = +\infty$ (see

(5.2)), and so,

$$h_i \leq (x_i \wedge b_i \setminus (ax))(ax)^{-1}\lambda = (P_H(x))_i(ax)^{-1}\lambda .$$

We deduce that (7.7) is valid for $k = i$.

Similarly, if $k \in \text{Argmax}(b, x)$, then by $b_k x_k = bx$ and $ax < bx$, we have $b_k x_k(bx)^{-1}\lambda = \lambda < b_k x_k(ax)^{-1}\lambda$, where by (7.6) we have $\lambda = b_k h_k \in \mathbb{R}$; whence $b_k \neq -\infty$. Consequently,

$$x_k(bx)^{-1}\lambda \leq h_k \leq x_k(ax)^{-1}\lambda . \quad (7.13)$$

Moreover, by (7.6), and the fact that $b_k \neq -\infty$, we get that $h_k = b_k \setminus \lambda$ for all $k \in \text{Argmax}(b, x)$, hence,

$$h_k = (b_k \setminus (ax))(ax)^{-1}\lambda .$$

This, together with the second inequality in (7.13), shows that

$$h_k = (x_k \wedge (b_k \setminus (ax)))(ax)^{-1}\lambda = (P_H(x))_k(ax)^{-1}\lambda$$

and so, (7.7) is valid for these k , which proves the claim.

Multiplying the second inequality in (7.7) by a_k , and using $P_H(x) \leq x$, we obtain that $a_k h_k \leq a_k x_k(ax)^{-1}\lambda \leq \lambda$ for all $k \in [n]$, and using (7.5), we get that $ah = \lambda$. Similarly, multiplying the second inequality in (7.7) by b_k , and using again $P_H(x) \leq x$, we obtain that $b_k h_k \leq b_k x_k(ax)^{-1}\lambda \leq bx(ax)^{-1}\lambda \leq \lambda$ for all $k \in [n]$, and using (7.6), we get that $bh = \lambda = ah$. This equality, together with (7.7), which is valid for all $k \in [n]$, imply 2°. ■

Remark 7.1. We observed in the proof of Theorem 7.1 that if h is an element of best approximation of x , the inequality (7.7) actually holds for all $k \in [n]$. It follows that

$$P_H(x)(bx)^{-1}\lambda \leq x(bx)^{-1}\lambda \leq h \leq P_H(x)(ax)^{-1}\lambda .$$

By comparing h with the extreme terms in the above inequalities, and using the characterization (3.14) of Hilbert's projective distance, we deduce that

$$d(h, P_H(x)) \leq (ax)^{-1}(bx) = d(x, H)$$

so that h lies in the intersection of two balls of radius $d(x, H)$ in Hilbert's projective metric, one being centered at the point x , the other being centered at the point $P_H(x)$.

Remark 7.2. One can give a geometric interpretation of the conditions of Theorem 7.1 in terms of faces of the ball with center x and radius $d(x, H)$. Indeed, let us fix some index $i \in \text{Argmax}(a, x)$, and let F_i denote the set of vectors h satisfying the conditions (7.5), (7.6), (7.7) of Theorem 7.1. Then the conditions that $a_i h_i = b_j h_j$ for all $j \in \text{Argmax}(b, x)$, together with $ax = a_i x$ and $bx = b_j x$, lead to $h_i h_j^{-1} = b_j a_i^{-1} = x_i x_j^{-1}(bx)(ax)^{-1}$. This can be rewritten with the usual linear algebraic notation, as

$$h_i - h_j = x_i - x_j + d(x, H), \quad \forall j \in \text{Argmax}(b, x).$$

Thus, if p is the cardinality of $\text{Argmax}(b, x)$, we see that h satisfies p of the inequalities defining the facets of the ball of radius $d(x, H)$ in Hilbert metric,

centered at the point x (the ball in Hilbert metric is a polyhedron in the usual sense, and so the standard notions of faces and facets -maximal faces-, see [23], apply to it). Therefore, the set F_i consisting of these vectors h lies in a $n-p$ dimensional face of this ball, and Theorem 7.1 gives a disjunctive representation of the set of elements of best approximation, as the union of the sets F_i with $i \in \text{Argmax}(i, x)$. Note that the inequalities (7.7) indicate that F_i may be a strict subset of a face of the latter ball, as illustrated in Figure 3 below (right).

Let us give some geometric interpretations of best approximation by closed half-spaces in simple particular cases.

Example 7.1. Let $n = 3$, and

$$H := \{h \in \mathbb{R}_{\max}^3 \mid h_2 \geq h_1\} = \{h \in \mathbb{R}_{\max}^3 \mid ah \geq bh\},$$

where $a = (-\infty, 0, -\infty)$, $b = (0, -\infty, -\infty)$, and let $x_1 > x_2 > x_3$. Then, with the notations of the proof of Theorem 7.1, we have $\text{Argmax}(a, x) = I = \{2\}$ and $\text{Argmax}(b, x) = J = \{1\}$, so necessarily $i = 2$ and $h = (h_1, h_2, h_3)^T$ is an element of best approximation of x in H if and only if there exists $\lambda \in \mathbb{R}$ such that

$$h_2 = h_1 = \lambda, \quad x_3 - x_1 + \lambda \leq h_3 \leq x_3 - x_2 + \lambda.$$

The half-space H was already represented in Figure 1, Assume now that $x = (2, 1, 0)^T$, so that, as noted in Example 6.1, $P_H(x) = (1, 1, 0)^T$. By Remark 7.2, the set of elements of best approximation is the set F_2 , which lies in a two dimensional face of a ball in Hilbert's metric. This set is represented by a bold segment in Figure 2.

Example 7.2. Consider now

$$H = \{h \in \mathbb{R}_{\max}^3 \mid \max(h_1, h_3) \geq h_2\}$$

and $x = (0, 1, 0)^T$. Here, $a = (0, -\infty, 0)$ and $b = (-\infty, 0, -\infty)$. We have $P_H(x) = (0, 0, 0)^T$, $d(x, P_H(x)) = 1$, and

$$\text{Argmax}(a, x) = \{1, 3\}, \quad \text{Argmax}(b, x) = \{2\}.$$

Theorem 7.1 shows that the set of elements of best approximation of x is the union of the sets F_1 and F_3 defined in Remark 7.2. Condition 3° of Theorem 7.1 yields

$$F_1 = \{h \in \mathbb{R}^3 \mid h_1 = h_2, -1 + h_1 \leq h_3 \leq h_1\}.$$

By symmetry, F_3 is obtained from F_1 by exchanging the variables h_1 and h_3 . This is illustrated in Figure 3 (left).

Example 7.3. Let

$$H := \{x \in \mathbb{R}_{\max}^3 \mid h_3 \geq \max(h_1, h_2)\}.$$

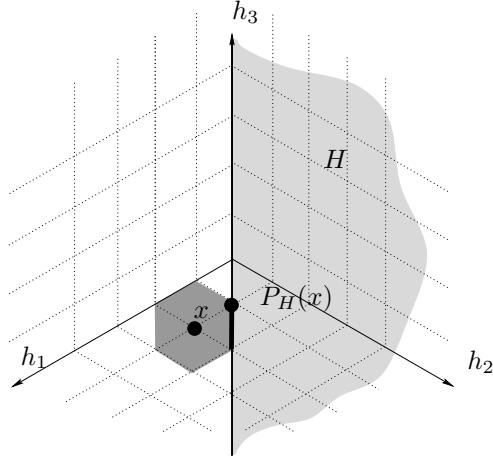


FIGURE 2. Illustration of Theorem 7.1 (see Example 7.1). The half-space $H = \{h \in \mathbb{R}_{\max}^3 \mid h_2 \geq h_1\}$ (light gray); the maximal open ball in Hilbert's metric centered at point $x = (2, 1, 0)^T$ and contained in the complement of H (dark gray): the projection $P_H(x)$ is visible at its boundary. The set of elements of best approximation of x is the bold segment.

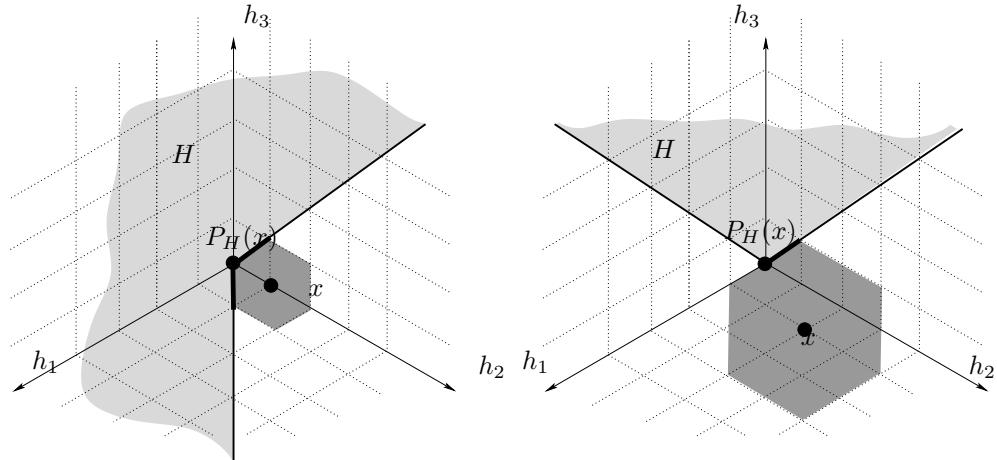


FIGURE 3. Left. A set of elements of best approximation of a disjunctive nature (Example 7.2). Right. The set of elements of best approximation may be a strict subset of a face of a Hilbert ball (Remark 7.2 and Example 7.3).

and $x = (1, 2, 0)^T$. It can be checked that $P_H(x) = (0, 0, 0)^T$, and that the set of elements of best approximation of x is a strict subset of a face of the ball of radius $d(x, P_H(x)) = 2$, centered at x , see Figure 3.

8. THE CYCLIC PROJECTION ALGORITHM TO SOLVE MAX-PLUS LINEAR SYSTEMS

The max-plus analogue, studied in [14], of the classical cyclic projection technique allows one to compute the canonical projection of a vector $u \in \mathbb{R}_{\max}^n$ onto a subsemimodule

$$V := V_1 \cap \cdots \cap V_p \quad (8.1)$$

defined as the intersection of p closed subsemimodules by successively projecting onto $V_1, V_2, \dots, V_p, V_1, \dots$. The application of this idea to the case of intersection of half-spaces, thanks to Theorem 5.1, will lead us to a new algorithm to solve the system of inequalities

$$Ax \geq Bx \quad (8.2)$$

where A, B are $p \times n$ matrices with entries in \mathbb{R}_{\max} .

Let us first explain how the method of [14] leads to a general algorithm. Formally, starting from an arbitrary finite vector $\xi^0 = u$, we compute the sequence

$$\xi^{k+1} = P_{V_{(k+1 \bmod p)}}(\xi^k) \quad \forall k \geq 0, \quad (8.3)$$

where $(l \bmod p)$ denotes the unique number belonging to the set $[p] = \{1, \dots, p\}$ congruent to l modulo p and P_{V_j} denotes the canonical projection onto V_j .

Theorem 8.1. *The sequence ξ^k generated by the cyclic projection algorithm is non-increasing and converges to $P_V(u)$.*

Proof. Since $P_{V_k}(x) \leq x$ holds for all x and for all k , we have

$$\xi^{k+1} = P_{V_{(k+1 \bmod p)}}(\xi^k) \leq \xi^k, \quad \forall k \geq 0,$$

so the sequence ξ^k is non-increasing. We prove by induction that

$$\xi^k \geq P_V(u), \quad \forall k \geq 0.$$

For $k = 0$, this follows from $u \geq P_V(u)$. Assume now that $\xi^k \geq P_V(u)$. Since $V \subset V_{(k+1 \bmod p)}$ and since $P_V(\cdot)$ is a monotone idempotent function, we have

$$\xi^{k+1} = P_{V_{(k+1 \bmod p)}}(\xi^k) \geq P_V(\xi^k) \geq P_V(P_V(u)) = P_V(u),$$

which concludes the proof by induction. Hence, the non-increasing sequence ξ^k must have a limit, ξ^∞ , such that

$$u \geq \xi^\infty \geq P_V(u).$$

Consequently, again since $P_V(\cdot)$ is a monotone idempotent function,

$$P_V(u) \geq P_V(\xi^\infty) \geq P_V(P_V(u)) = P_V(u),$$

whence $P_V(u) = P_V(\xi^\infty)$. Therefore, in order to show that the equality $\xi^\infty = P_V(u)$ holds, it suffices to show that $\xi^\infty = P_V(\xi^\infty)$, i.e., that $\xi^\infty \in V$.

Observe that for all $m \in [p]$, ξ^∞ is a limit of the subsequence of ξ^k obtained by taking all the indices k such that $(k + 1 \bmod p) = m$. Since V_m is closed, it follows that $\xi^\infty \in V_m$. Since this holds for all $m \in [p]$, we deduce that $\xi^\infty \in V$. ■

The following is an immediate corollary.

Corollary 8.1. *The intersection $V = V_1 \cap \dots \cap V_p$ is not reduced to the $-\infty$ vector if and only if the cyclic projection algorithm, initialized by taking ξ^0 to be any finite vector u , converges to a non- $(-\infty)$ vector ξ^∞ (and then this vector is precisely $\xi^\infty = P_V(u) \in V$). ■*

Applying this algorithm to the case of intersection of half-spaces, and using Theorem 5.1, we obtain the following algorithm to solve the system of inequalities (8.2), where A, B are $p \times n$ matrices with entries in \mathbb{R}_{\max} . We have

$$V = H_1 \cap \dots \cap H_p, \quad (8.4)$$

where H_j is the half-space

$$H_j := \{x \in \mathbb{R}_{\max}^n \mid A_j x \geq B_j x\} \quad \forall j \in [p], \quad (8.5)$$

with $A_j := (A_{j1}, \dots, A_{jn})$ and $B_j := (B_{j1}, \dots, B_{jn})$ denoting the j th rows of A and B , respectively. Hence, by Theorem 5.1, we obtain

$$P_{H_j}(x) = x \wedge (B_j \setminus A_j(x)) \quad \forall x \in \mathbb{R}_{\max}^n, \forall j \in [p], \quad (8.6)$$

and thus, in particular,

$$\xi^{k+1} = P_{H_j}(\xi^k) = \xi^k \wedge (B_j \setminus A_j(\xi^k)) \quad \forall k \geq 0, \quad j := (k + 1 \bmod p).$$

Componentwise this means, by (1.8) and (5.2), that for each $k = 0, 1, \dots$ we have

$$\xi_i^{k+1} = P_{H_j}(\xi_i^k) = \xi_i^k \wedge (B_{ji} \setminus (A_j(\xi^k))) \quad \forall i \in [n], \quad (8.7)$$

where again $j = (k + 1 \bmod p)$ and $A_j(\xi^k) = \bigoplus_{i=1}^n A_{ji} \xi_i^k$.

An alternative method to the cyclic projection technique is the following *power algorithm*, which is based on the observation that $Ax \geq Bx$ if and only if $x = B^\# Ax \wedge x$ (see (2.33)). The latter fixed point problem can be solved by the iterative scheme

$$\eta^0 = u, \quad \eta^{k+1} = B^\# A \eta^k \wedge \eta^k, \quad \forall k \geq 0, \quad (8.8)$$

with $B^\#$ of (2.30). This method may be thought of as a generalization of the *alternated projection algorithm* of Butković and Cuninghame-Green [11] which concerns the special case of the linear system $Ay = Bz$ (the latter can be reduced to the former by setting $x = (y, z)$ and suitably extending the matrices A and B).

In order to compare the power algorithm with the cyclic projection algorithm we shall need the following “sandwich theorem”:

Theorem 8.2. Consider the linear system $Ax \geq Bx$. Let η^k and ξ^k denote the sequences generated by the power and cyclic projection algorithms, respectively, initialized with the same initial condition u . Then

$$P_V(u) \leq \xi^{pk} \leq \eta^k \quad \forall k \geq 0 . \quad (8.9)$$

Proof. We shall use that the operator $\eta \rightarrow (B^\# A\eta)_j \wedge \eta_j$ is monotone.

The first inequality follows from Theorem 8.1.

We now show that $\xi^p \leq \eta^1$. By (2.30) we have

$$\eta_j^1 = (B^\# Au)_j \wedge u_j = \wedge_{l=1}^p (-B_{lj} +' (A_l u)) \wedge u_j = (-B_{ij} +' (A_i u)) \wedge u_j,$$

for some $i \in [p]$. Hence, using that ξ^k is non-increasing, (8.7) for $k = 0$ and (5.2), it follows that

$$\xi_j^p \leq \xi_j^i = (B_i \setminus (A_i \xi^{i-1}))_j \wedge \xi_j^{i-1} \leq (B_{ij}^{-1} A_i u) \wedge u_j = \eta_j^1.$$

The inequality $\xi^{pk} \leq \eta^k$ is obtained by induction. For $k = 1$ it is already proved. Assume now that it holds for k replaced by $k - 1$. Then, by (2.30) we have

$$\begin{aligned} \eta_j^k &= (B^\# A\eta^{k-1})_j \wedge \eta_j^{k-1} = \wedge_{l=1}^p (-B_{lj} +' (A_l \eta^{k-1})) \wedge \eta_j^{k-1} \\ &= (-B_{ij} +' (A_i \eta^{k-1})) \wedge \eta_j^{k-1}, \end{aligned}$$

for some $i \in [p]$. Then using that ξ^k is non-increasing and

$$\xi^{p(k-1)+i-1} \leq \xi^{p(k-1)} \leq \eta^{k-1},$$

it follows that

$$\begin{aligned} \xi_j^{pk} &\leq \xi_j^{p(k-1)+i} = (B_i \setminus (A_i \xi^{p(k-1)+i-1}))_j \wedge \xi_j^{p(k-1)+i-1} \\ &\leq (B_{ij}^{-1} A_i \eta^{k-1}) \wedge \eta_j^{k-1} = \eta_j^k. \end{aligned}$$

The correctness of the power algorithm follows from the next result.

Theorem 8.3. The sequence η^k produced by the power algorithm initialized with $\eta^0 = u$ is non-increasing and converges to $P_V(u)$. Moreover, if u has finite integer entries, if V contains at least one finite vector, and if all the entries of the matrices A, B belong to $\mathbb{Z} \cup \{-\infty\}$, then, $\eta^m = P_V(u)$ for all $m \geq n \times d(x, V)$.

Proof. By (8.8) and (8.9), we have $\eta^k \geq \eta^{k+1} \geq P_V(u)$ ($k = 0, 1, \dots$). Hence the non-increasing sequence η^k must have a limit, η^∞ , such that $u \geq \eta^\infty \geq P_V(u)$. To show that the equality $\eta^\infty = P_V(u)$ holds, by the definition of P_V it suffices to show that $\eta^\infty \in V$. But, passing to the limit for $k \rightarrow \infty$ in (8.8) we obtain

$$\eta^\infty = B^\# A\eta^\infty \wedge \eta^\infty,$$

whence by (2.33), it follows that $A\eta^\infty \geq B\eta^\infty$, that is, $\eta^\infty \in V$.

Assume now that the conditions of the second part of the theorem hold, and let v denote a finite vector in V . Then, $u \geq v\lambda$, for some finite scalar λ , and so $P_V(u) \geq v\lambda$ is finite.

Moreover, we already showed that $P_V(u) = \eta^\infty$ is the limit of the sequence of vectors η^k , and it follows from the construction of this sequence in (8.8) that for all k , the entries of η^k belong to $\mathbb{Z} \cup \{-\infty\}$, as soon as the entries of A, B and u do. Therefore, $P_V(u) \in (\mathbb{Z} \cup \{-\infty\})^n$, and since we observed that $P_V(u)$ is finite, we must have $P_V(u) \in \mathbb{Z}^n$. Moreover, $\eta^k \in \mathbb{Z}^n$ since $\eta^k \geq P_V(u)$.

We claim that

$$P_V(u) \setminus u = 0 . \quad (8.10)$$

Indeed, the inequality $P_V(u) \setminus u \geq 0$ follows from $P_V(u) \leq u$. If we had $P_V(u) \setminus u > 0$, then, we would have $P_V(u)\lambda \leq u$ for some $\lambda > 0$, but then the vector $w := P_V(u)\lambda > P_V(u)$ would be such that $w \in V$ and $w \leq u$, contradicting the definition of $P_V(u)$ as the maximal element with the latter properties. This proves (8.10).

Hence,

$$d(u, V) = d(u, P_V(u)) = ((u \setminus P_V(u))(P_V(u) \setminus u))^- = (u \setminus P_V(u))^- .$$

Since u and $P_V(u)$ are finite vectors, $u \setminus P_V(u)$ is finite, and so, using (1.3), we deduce from

$$u(u \setminus P_V(u)) \leq P_V(u)$$

that

$$u \leq P_V(u)(u \setminus P_V(u))^- .$$

Hence,

$$P_V(u) \leq u \leq P_V(u)d(u, V) .$$

In order to analyze the complexity of the algorithm, we return to the usual notation for the addition, and consider the function from \mathbb{Z}^n to \mathbb{Z} ,

$$E(\eta) := \sum_{i \in [n]} (\eta_i - (P_V(u))_i).$$

Observe that $E(\eta) \geq 0$ for all $\eta \geq P_V(u)$. Moreover, if $\eta^m = \eta^{m+1}$, then, $\eta^k = \eta^m$ must hold for all $k \geq m$, and so, $\eta^m = \lim_k \eta^k = P_V(u)$. In addition, if m is the smallest index such that $\eta^m = \eta^{m+1}$, then, the sequence of integer vectors η^0, \dots, η^m is strictly decreasing. In particular, at every step $k < m$, there is at least one coordinate $i \in [n]$ such that $\eta_i^k > \eta_i^{k+1}$. Thus,

$$n \times d(u, V) \geq E(\eta^0) > E(\eta^1) > \dots > E(\eta^m) = 0 .$$

Since $E(\eta^0), \dots, E(\eta^m)$ are integers, we deduce that $m \leq n \times d(u, V)$. ■

Corollary 8.2. *The intersection $V = V_1 \cap \dots \cap V_p$ is not reduced to the $-\infty$ vector if and only if the power algorithm, initialized by taking η^0 to be any finite vector u , converges to a non- $(-\infty)$ vector η^∞ (and then this vector is precisely $\eta^\infty = P_V(u) \in V$). ■*

The power algorithm (8.8) can be rewritten componentwise as

$$\eta_i^{k+1} = (B_i^\sharp(A\eta^k)) \wedge \eta_i^k, \quad \forall i \in [n], \quad \forall k \geq 0 . \quad (8.11)$$

This should be compared with the cyclic projection algorithm for $Ax \geq Bx$, that is, (8.7). Note that one step of the power algorithm requires $O(m)$ operations, where m is the total number of finite entries in the matrices A and B , whereas step i of the cyclic projection algorithm only requires $O(m_i)$ operations, where m_i is the total number of finite entries of the rows A_i and B_i . Since ξ^k and η^k decrease to the same limit, $P_V(u)$, the “sandwich” theorem 8.2 shows that the cyclic projection algorithm is always at least as fast as the power algorithm, since for the same effort of computation, it produces a closer upper bound of $P_V(u)$. Indeed, computing ξ^{pk} requires an $O(k(m_1 + \dots + m_p)) = O(km)$ time, and computing η^k also requires an $O(km)$ time.

Example 8.1. The following example shows that the cyclic projection algorithm may yield a speedup by a factor n , by comparison with the power algorithm.

Consider the system of $n - 1$ inequations in n variables:

$$x_1 \leq -1 + x_n, \quad x_2 \leq -1 + x_1, \dots, \quad x_{n-1} \leq -1 + x_{n-2} .$$

When $n = 6$, this corresponds to the following 5×6 matrices

$$B = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix} \quad A = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix} ,$$

where $-\infty$ is represented by the “.” symbol.

The cyclic projection algorithm, initialized with the zero vector, yields the sequence

$$\begin{aligned} \xi^0 &= (0, \dots, 0)^T \\ \xi^1 &= (-1, 0, 0, \dots, 0)^T \\ \xi^2 &= (-1, -2, 0, \dots, 0)^T \\ &\vdots \\ \xi^{n-1} &= \xi^n = (-1, -2, \dots, -(n-1), 0)^T . \end{aligned}$$

The algorithm converges in n steps, and every step takes $O(1)$ operations, which makes a total of $O(n)$ operations. Indeed, note that every row B_j and every row A_j have $O(1)$ entry equal to $-\infty$, which implies that the update of ξ can be done in only $O(1)$ time.

The power algorithm, initialized with the same vector, yields the sequence

$$\begin{aligned}\eta^0 &= (0, \dots, 0)^T \\ \eta^1 &= (-1, -1, \dots, -1, 0)^T \\ \eta^2 &= (-1, -2, -2, \dots, -2, 0)^T \\ \eta^3 &= (-1, -2, -3, -3, \dots, -3, 0)^T \\ &\vdots \\ \eta^{n-1} &= \eta^n = (-1, -2, \dots, -(n-1), 0)^T .\end{aligned}$$

The algorithm also converges in n steps, but every step now takes an $O(n)$ time, since computing every coordinate of $B \setminus (A\eta)$ requires a $O(1)$ time. Thus, the power algorithm requires a total of $O(n^2)$ operations, and the cyclic projection algorithm shows a speedup of n .

In this example, the matrices are very sparse. One readily gets an example of full matrices with the same speedup by replacing every $-\infty$ entry by a value close enough to $-\infty$, which will not modify the sequences produced by the cyclic projection and by the power algorithm. Now, every step of the cyclic projection algorithm takes a $O(n)$ time, and every step of the power algorithm requires a $O(n^2)$ time. Hence, we keep a speedup of n .

Remark 8.1. Theorem 8.3 gives a bound for the convergence time of the power and cyclic projection algorithms which is pseudo-polynomial, meaning that the convergence time is bounded by a polynomial expression in the integers constituting the input of the problem. To see this, let us recall the explicit expression of the projector, from [8],

$$P_V(u) = \sup_{i \in I} v_i(v_i \setminus u) ,$$

where $(v_i)_{i \in I}$ is an arbitrary generating family of V . A canonical choice of the generating family consists of representatives of the extreme rays of V ; then, the explicit bound in [3, Proposition 10] shows that the finite entries of the vectors v_i , and so, $d(u, V)$, are polynomially bounded in terms of the finite entries of the matrices A and B .

Remark 8.2. The following simple example shows that the convergence time of both algorithms is actually only pseudo-polynomial. Assume that V is defined by the inequalities $x_1 \leq \max(0, -1 + x_2)$, $x_2 \leq x_1$, and let us initialize both algorithms with $u = (k, k)^T$, so that $(0, 0)^T = P_V(u)$ and $d(u, P_V(u)) = k$. Then, it can be checked that both algorithms take k iterations to converge, whereas for a polynomial time algorithm, a number of iterations polynomial in $\log k$ would be required. Let us note in this respect that the problem of solving systems of max-plus inequalities is equivalent to solving mean payoff games (see [20, 1]), and that the existence of a polynomial time algorithm for mean payoff games is an open question.

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